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Gauge-Gravity Dualities, Dipoles and New Non-Kähler Manifolds

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Abstract

In this work we explore many directions in the framework of gauge-gravity dualities. In type IIB theory we give an explicit derivation of the local metric for five branes wrapped on rigid two-cycles. Our derivation involves various interplays between warp factors, dualities and fluxes and the final result confirms our earlier predictions. We also find a novel dipole-like deformation of the background due to an inherent orientifold projection in the full global geometry. The supergravity solution for this deformation takes into account various things like the presence of a non-trivial background topology and fluxes as well as branes. Considering these, we manage to calculate the precise local solution using equations of motion. We also show that this dipole-like deformation has the desired property of decoupling the Kaluza-Klein modes from the IR gauge theory. Finally, for the heterotic theory we find new non-Kähler complex manifolds that partake in the full gauge-gravity dualities and study the mathematical structures of these manifolds including the torsion classes, Betti numbers and other topological data.

Contents

1. Introduction and Summary

It is by now clear that the usual way to deal with flux compactifications is to replace the Calabi-Yau $SU(n)$ holonomy condition with an $SU(n)$ or $SU(n{-}p)$ structure condition, with p being a specific integer [1], [2], [3]. This requires, for example, the existence of two globally defined spinors on the six-dimensional manifold which are everywhere parallel for the $SU(3)$ structures and nowhere parallel for the $SU(2)$ structure (here $p = 1$). These spinors appear in the supersymmetry transformations of the gravitino and dilatino fields and as a result the supersymmetry conditions drastically restrict the fluxes and the geometry. The restrictions on the geometry are seen in terms of the torsion classes which measure the departure from the Calabi-Yau condition. The $SU(3)$ structure is much more tractable as there are only five torsion classes whereas for the $SU(2)$ structure the torsion decomposes into ninety classes which makes the classification a formidable task [4].

An alternative route by which we retain the properties of $SU(n)$ structures and their consequent classification in terms of torsion classes, yet do not explicitly consider the supersymmetry transformation, is to follow a U-duality map that appears directly from superstring compactifications. The beauty of this approach is that it gives solutions that are explicitly supersymmetric, satisfy the required equations of motion *and* fall under the classification of torsion classes for $SU(n)$ structure, all in one smooth map. Such an approach was first elucidated in [5], and was later followed in various other works, for example [6], [7], [8], [3], [9], and [10] to name a few. The paper [5] also pointed out some smooth examples which were explicitly verified and studied recently in [11].

Most of these other examples were in the $SO(32)$ heterotic theory where the compact six-dimensional manifold was a non-Kähler complex manifold with an $SU(3)$ structure with generic torsion classes given by

$$\mathcal{W}_1 \;=\; \mathcal{W}_2 \;=\; 0, \qquad \mathcal{W}_3 \;\neq\; 0, \qquad 2\mathcal{W}_4 \;+\; \mathcal{W}_5 \;=\; 0, \tag{1.1}$$

where the first condition implies complexity, the second implies non-zero torsion and the third implies supersymmetry [3], [12]. Recall that the mathematical structure of torsion classes was developed sometime earlier in [1], [2] and the examples coming from string theory were the first concrete realisation of manifolds having $SU(3)$ structures. A somewhat similar classification was also given for manifolds with G_2 structure in [2]; elaborating on earlier works of [13], [14], [15]. Further geometrical and topological properties of these manifolds appeared in [16], [9], [17], and [18].
All the examples mentioned above were compact complex manifolds and therefore the next venture was to look for manifolds that are:

- Compact, non-complex and non-Kähler
- Non-compact, (non)-complex and non-Kähler

Examples of the first kind were easy to come by and were constructed explicitly in [19] for heterotic theories, [20], [21] for type IIB theories and in [22] for type IIA theories (plus many other subsequent papers). In all these cases moduli stabilisation went hand in hand with flux compactifications. The physical picture was also clear: all these manifolds correspond to four-dimensional compactifications with almost zero moduli.

Of course such a physical picture cannot be provided for the second of the above examples, because non-compactness of the internal manifold will imply zero Newton's constant in four-dimensions. So any physical picture has to come from a different consideration. This is where the idea of gauge-gravity dualities became useful in realising some of the above manifolds. The dualities that we are looking for cannot be between CFTs and gravity, but have to be between theories with non-zero beta function and gravity. We already know of one concrete example in type IIB: the Klebanov-Strassler solution [23] that provides the gravity duals of cascading gauge theories. However the underlying manifolds therein are non-compact, but complex and (conformally)-Kähler manifolds. Another example in type IIB theory is the Maldacena-Nunez solution [24] which is a non-compact, complex, non-Kähler manifold; but it only gives the IR of the gauge theory. The UV completion of this is addressed in [25]. Therefore now the question is: can we construct examples of these manifolds in other string theories like Type IIA, heterotic and Type I?

The answer to this question was given in the affirmative by [26], [27] and [28] (see also [29] for a short review). The solutions that we provided in type IIA turned out to be explicitly non-complex and non-Kähler and formed gravity duals of wrapped D6 branes on three-cycles of another non-complex non-Kähler manifold. On the other hand the heterotic/Type I examples were complex manifolds. The non-compactness of these manifolds was necessary to allow the charges of the wrapped branes to escape.

One important thing about our approach was that we were not necessarily concerned with checking the supersymmetry transformation, even though we argued the existence of SUSY for each of the solutions we presented. This is of course in the same vein as our earlier discussion, and in fact the U-duality map here turned out to be nothing other than the mirror symmetry itself realised as a Strominger-Yau-Zaslow map [30] along with a series of flops and dimensional reductions. This duality map connected *all* the string theories together in a cycle. It also turned out to be a powerful solution-generating technique, wherein a starting type IIB solution when cycled through the duality map gave consistent solutions in all other theories.

The starting type IIB solution was realised as an F-theory configuration. This guaranteed one immediate benefit: the resulting type IIB solution was naturally supersymmetric, and hence satisfied equations of motion. Therefore without offering a firm classification in terms of $SU(3)$ or $SU(2)$ structure we got our required supersymmetric solution. However, since F-theory is a much more involved scenario, the resulting type IIB solution had additional local and non-local seven-branes. This made the determination of the full global metric very difficult. At this point we could do two things: one, we could ignore the F-theory picture altogether and try to determine directly the type IIB solution, or two, consider only a local picture in type IIB.

An attempt to determine directly the type IIB solution turned out to give only a non-supersymmetric background metric (see [31], [32], [26], [28]). It wasn't clear whether there would exist a supersymmetric background metric with the required properties at all. Recall that the background of [24] had only one kind of three-form flux H_{NS} or H_{RR} and the metric was non-Kähler. Our aim in type IIB was to get a conformally Kähler metric with both H_{NS} *and* H_{RR} simultaneously. Therefore instead of trying out an explicit solution directly in type IIB, we tried the next available choice: an F-theory picture [26], [27], [28].

Although this choice didn't help us to get the full global metric, it provided us with a consistent local picture. In fact the F-theory picture turned out to be richer than we expected. One thing was clear from the very beginning, namely the full global metric in type IIB can be constructed to be conformally Kähler irrespective of whether we know the actual metric or not! The discussion of why a local metric may suffice in the story will be discussed in sec 2.1. A surprising output of all these was that we got *fundamental* matter for free [28]! In fact if we now move the fundamental matters away from our local region, then we can in principle simulate a pure $\mathcal{N} = 1$ gauge theory in the IR. Thus F-theory provides us with a nice solution of the dilemma. A more detailed discussion of this will be presented in sec. 2.2.

The local metric that we presented earlier in [26], [28], [27] was derived using various resources. They all lead to similar results. In sec 2 we will rederive the whole thing using strict equations of motion. There it will be clear how various warp factors conspire to give us the right result. In fact a detailed analysis of an order-by-order expansion of the warp factor will show us a nice and consistent emerging picture that strongly supports our earlier result.

In the local picture, the situation is almost identical to the one of [33] where a $\mathbf{T}^6$ torus with NS fluxes was mapped into a half-flat manifold. The model of [33] doesn't solve the equations of motion, but is simple enough to clarify the story. On the other hand, the way we describe it is to start with a Kähler manifold (not necessarily Calabi-Yau but with a non-trivial $\mathbf{P}^1$ on which we can have wrapped D5 branes) and add NS fluxes which determine a nonzero $\mathcal{W}_3$ torsion class. The next step is to add local and non-local seven branes turning on other torsion classes. If the departure from the orientifold point is adiabatic, the manifold will be a small deformation of a half-flat manifold which would then share the nice properties of being expressible in terms of the harmonic functions, as discussed in [33].

The NS fluxes that correspond to turning on some torsion classes in the geometry also have two possible inter-relating impacts on the field theory. First, these NS fluxes give rise to *dipole* deformations in the field theory. These dipoles are somewhat different from the ones studied earlier in [34]. Our scenario is more involved than the one considered in [34] as we have non-trivial background topology, branes and fluxes. A detailed discussion of this is given in sec 4.1. Secondly, from our computations, we can make a strong statement concerning the masses of the KK modes in the presence of the NS field. We explicitly show that the KK modes are heavier in the presence of the NS fluxes which for confining theories was first observed in the second paper of [35]. Being explicit, our solution does not have the potential problems detailed in [36]. Our solution represents a concrete example of a dipole deformed field theory, even though for all IR effects (that we are mostly concerned with) the dipole deformations are not visible. Elaborating on the above analysis will be the subject of sec. 4.2.

Once we are in the realm of non-complex, non-Kähler manifolds, we should also entertain possibilities of having *generalised* complex structures *a là* Hitchin-Gualtieri [37], [38]. Our type II constructions have all the necessary ingredients to realise these new configurations in string theory. Some earlier attempts to discuss the *algebraic* aspects of these manifolds have been presented in [39], [40]. But a full supergravity analysis of these new constructions still awaits a thorough treatment. We will not attempt this here, and a more detailed analysis on this will be presented soon in [41].

Parallel to these developments are similar considerations for the heterotic theory. We have already constructed examples that might indicate possible gauge-gravity dualities also in heterotic theory [27], [28]. In [28] we gave an example of a manifold without branes but only with fluxes (or *torsion* here) that might form a gravity dual of the theory on heterotic NS5 branes. The metric was a complex non-Kähler and non-compact manifold that had some resemblance with the metric of [24]. In sec. 3.3 we will discuss some new non-Kähler manifolds, some of which give rise to the metric *before* geometric transition, i.e metrics on which we can have wrapped five-branes. We will be able to provide detailed discussion on the topological properties of these manifolds, including a determination of the Betti numbers and the cohomology classes.

Having such explicit background solutions with and without branes still do not provide a convincing proof that they are related by some gauge-gravity duality. We need more detailed analysis. One possible way to in-principle confirm this is via a topological theory argument, much like the one presented for type IIB [42]. As we know, there is no "standard" topological theory or topological twist in the heterotic theory. There is only a *half-twist* [43] and therefore one would need to ask in the half-twisted theories whether we can have dualities like [42]. Existence of such a scenario would confirm possibilities of gauge-gravity dualities in the heterotic theory. A recent attempt to study correlation functions in (0,2) theories has been addressed in [44], [45]. But to say something concrete, more work is needed.

Finally one might also want to study *twisted* generalized complex structures for a system like ours. An earlier work is [46], done for simple cases. For a background with non-trivial topology a twisted version of a generalised complex structure may be a hard problem to trace directly from a sigma-model point of view but on the other hand knowing the explicit local geometry might shed some light here. These aspects are still under investigation.

1.1. Formal outline of the paper

In this paper we have tried to discuss many new aspects of gauge-gravity dualities. Some of these aspects correspond to our earlier studied models of geometric transitions. In sec. 2 we give a detailed derivation of the local supersymmetry-preserving metric with branes and fluxes, from equations of motion. The emphasis of sec. 2.1 is to elucidate the non-trivial nature of the $U(1)$ fibration of the local metric. We show how warp factors and fluxes conspire to give the right fibration. This analysis is done without considering all the back-reactions of branes etc. In sec. 2.2 we study a fully supersymmetric configuration with D5s, D7s and fluxes and their back-reactions. We give the possibility of the existence of bound states of D5s on a single D7 brane that could potentially occur at a point in the moduli space of our configuration.

Section 3 is mostly a study of the corresponding heterotic picture. The heterotic story that we present here (that has also appeared in some of our earlier works [27], [28]) is not in any way *dual* to the type IIB background. Although the original derivation was motivated by some duality arguments (see [27]), the final configuration is deformed away from the original result to a new metric that satisfies equations of motion and is supersymmetric. The deformation is non-adiabatic and so cannot be realised as a perturbation. As discussed earlier in [27], [28] there are two different heterotic backgrounds conveniently classified as *before* and *after* geometric transition. In [28] we analysed the background after geometric transition. In sec. 3.1 we study the background before geometric transition. This is a background given in terms of non-trivial NS5 branes wrapped on a two-cycle of the geometry. Our analysis show that the resulting manifold is a new non-compact, non-Kähler manifold that could even be complex. In this section we manage to provide the local metric of the manifold, and in sec 3.2 we study the torsion classes associated with this manifold. The manifold(s) that we find are new, and in sec. 3.3 we study a family of such manifolds including their mathematical structures and Betti numbers.

The analysis that we present in sec. 2 took all the branes and fluxes into account. This analysis, however, could be thought of as though we are *away* from the orientifold point. At the orientifold point we have to carefully take the projections, and this allows only some special B_{NS}, B_{RR} fields. The choice of these fields tells us that we have a *dipole* deformation in the field theory. In sec 4 we elucidate this in great detail. Due to non-trivial topology, branes and fluxes, the analysis turns out to be particularly involved, and so we do this in two steps. Step one is sec 4.1 where we study the system without incorporating branes but keeping only non-trivial fluxes and topology. Step two is sec. 4.2 wherein we put all the branes in the geometry and study the possible deformations. With some effort we manage to calculate the precise local metric with dipole deformation[1]. We also find something very interesting: the decoupling of KK modes on the wrapped D5 branes. The two-cycle on which we have wrapped D5 branes *shrinks* in volume due to the background dipole deformation. We show this by evaluating the volume both before and after deformation, and then calculating the difference.

Finally in sec. 5 we give a short discussion and point out possible future directions.

2. New results on geometric transitions in type IIB theory

This is a further continuation of our works [26], [27] and [28], but now we would like to address issues like solving equations of motion, possible dipole deformations and new non-Kähler manifolds. Our earlier works were basically elaborating the story of geometric transitions by constructing precise supergravity solutions that could be used to study the gauge/gravity dualities more consistently. Recall that prior to our papers [26], [27] and [28], there were *no* supergravity descriptions for geometric transitions. Most of the earlier descriptions were based on topological identifications that started off with [42] (see also [47]) and were soon incorporated in string theory by [48], [49], [50]. Although many new developments were reported using these identifications, a precise supergravity description was called for so that an explicit quantitative analysis could be performed. This was not a concern for the other two parallel developments of [23] and [24] that explored the same scenario from a sightly different angle because of the existence of supergravity backgrounds that formed the duals of cascading confining gauge theories.

[1] Up to some possible subtleties that we will mention as we go along.

2.1. Precise supergravity analysis

Our analysis of type IIB started with the dual of $\mathcal{N} = 1$ $SU(M)$ gauge theory that forms the far IR of a gauge theory whose UV description involves $SU(N) \times SU(N+M)$ gauge fields with two different coupling constants[2]. In the gravity description the far UV picture is captured by going to large radial distances i.e $r \to \infty$ in a non-compact Kähler geometry. The IR picture, on the other hand, is captured when r is small and this is also dual to pure $\mathcal{N} = 1$ gauge theory. Clearly the UV description can become complicated by various factors:

• Existence of flavors: Flavors can drastically modify the UV picture for our case. In the model studied in [28], we showed that there are two different flavors that could be considered in the story − fundamental and bi-fundamental − which partake in the full UV description. The fundamental flavors come from the seven-branes (not necessary all local) and the bi-fundamental flavors come from the three-branes (necessarily local). At low energies these flavors are either massive or reduce in number by a renormalisation group flow and Seiberg dualities. At high energies they can modify the story in interesting ways, so we need to consider them carefully.

• Existence of KK modes: One of the clear distinctions between the geometric transition picture and the Klebanov-Strassler model is the existence of KK modes in the UV for the former case. Recall that the UV of the geometric transition is a *six* dimensional theory whereas the Klebanov-Strassler model remains four-dimensional throughout. Once the theory becomes six-dimensional i.e. the effect of the $\mathbf{P}^1$ starts showing up, we have to consider the full theory on the wrapped D5 branes. This would mean that from a four-dimensional point of view we have to take into account the full tower of KK states on the sphere. This becomes a formidable problem.

Because of these issues, we see that the full $r \to 0$ to $r \to \infty$ geometry is complicated. In the Klebanov-Strassler case many of the above issues could be avoided, so a global geometry can be easily considered. For our case we cannot ignore the KK modes, so this will definitely make the UV behavior different from the Klebanov-Strassler case. What about the flavors? Again, clearly the bi-fundamental matter is the core of the story and could not be avoided (even in the Klebanov-Strassler model), so the question would be

[2] There could be subtleties associated with the existence of Baryonic branches that take us to different IR theories [25], [51]. For our case we will ignore them here as we are only concerned with $\mathcal{N} = 1$ $SU(M)$ IR theories. Details on the other cases will be in the sequel to this paper.

regarding the fundamental matter. As discussed above, these are given by the local and non-local seven branes. So can we ignore these seven-branes as we did for the Klebanov-Strassler solution?

The model that we constructed in [26], [27] and [28] required us to take an F-theory solution which is a four-fold that could be considered as a T^2 fibration over a Kähler base $\mathcal{B}$. The base $\mathcal{B}$ has at least one $\mathbf{P}^1$ that is topologically non-trivial. On this two-cycle we can wrap M D5 branes and this can easily give us $\mathcal{N}=1$ $SU(M)$ gauge theory. However the F-theory torus also has to degenerate on the base, and this will give us local and non-local seven-branes.

Having an underlying F-theory solution serves multi-fold purposes. It can easily give us a type IIB background that preserves supersymmetry in the presence of fluxes and branes. The base of the fourfold can be made compact or non-compact; Kähler or non-Kähler. In all cases we can have gauge theories preserving minimal supersymmetry. The fourfold that we constructed in [28] had a Kähler base in the absence of branes and fluxes. In the presence of fluxes and branes we know that the base could become conformally Kähler or even non-Kähler. One might then expect that the overall metric can be written as

$$\begin{aligned} ds^2 = F_0(\tilde{r})\ ds^2_{0123} + F_1(\tilde{r})\ d\tilde{r}^2 + F_2(\tilde{r})\ (d\tilde{\psi} + \cos\tilde{\theta}_1 d\tilde{\phi}_1 + \cos\tilde{\theta}_2 d\tilde{\phi}_2)^2 + \\ + \left[F_3(\tilde{r})\ d\tilde{\theta}_1^2 + F_4(\tilde{r})\ \sin^2\tilde{\theta}_1 d\tilde{\phi}_1^2\right] + \left[F_5(\tilde{r})\ d\tilde{\theta}_2^2 + F_6(\tilde{r})\ \sin^2\tilde{\theta}_2 d\tilde{\phi}_2^2\right], \end{aligned} \tag{2.1}$$

where the $F_i(\tilde{r})$ are the warp factors and we have labelled the coordinates of the fourfold base as the radial coordinate $\tilde{r}$, the two spherical coordinates $(\tilde{\theta}_i, \tilde{\phi}_i)$ and the $U(1)$ fibration as $\tilde{\psi}$.

There are a few important details regarding the above solution that we should mention at this stage. First of all observe that we have used *global* coordinates to write it. That would mean that this solution is valid at $\tilde{r} \to \infty$ also. Whether or not this could be the case still remains to be seen, because our analysis from F-theory was done without considering localised three-branes. Thus (2.1) will have to be changed to reflect the local behavior only.

Secondly, the way we constructed the metric tells us that the background explicitly preserves supersymmetry. Thus this solution is *different* from the one proposed in [31] as we discussed in great detail in [26], [27] and [28]. The only region where the two metrics

(2.1) and the one in [31] look similar in form is locally. The local behavior of both metrics gives us

$$ds^2 = dr^2 + (dz + \Delta_1^0 \text{cot } \langle\theta_1\rangle\ dx + \Delta_2^0 \text{cot } \langle\theta_2\rangle\ dy)^2 + (d\theta_1^2 + dx^2) + (d\theta_2^2 + dy^2) \quad (2.2)$$

Here $(r, z, x, \theta_1, y, \theta_2)$ are the local coordinates measured from a chosen point $\mathbf{P}_0$ in our six-dimensional space (2.1), where

$$\mathbf{P}_0 \ = \ r_0, \langle\psi\rangle, \langle\phi_1\rangle, \langle\theta_1\rangle, \langle\phi_2\rangle, \langle\theta_2\rangle \quad (2.3)$$

with Δ_i^0 in the above local metric defined as

$$\Delta_1^0 \ = \ \sqrt{\frac{F_2(r_0)}{F_4(r_0)}}, \qquad \Delta_2^0 \ = \ \sqrt{\frac{F_2(r_0)}{F_6(r_0)}}. \quad (2.4)$$

Imagine now that we choose our point $\mathbf{P}_0$ not in the space (2.1), but at a point in the space with the metric of [31]. How does the behavior of Δ_i^0 change? One can easily show by solving the background equations of motion that the behavior of Δ_i^0 is now:

$$\Delta_1^0 \ = \ \sqrt{\frac{\gamma_0'}{\gamma_0}}\ r_0, \qquad \Delta_2^0 \ = \ \sqrt{\frac{\gamma_0'}{\gamma_0 + 4a^2}}\ r_0 \quad (2.5)$$

with a^2 being the resolution factor. Thus up to re-definitions of Δ_i^0 the local behaviors are exactly identical!

There are still a few loose ends that we need to clarify before moving ahead. All have to do with our metric (2.1) and its local version (2.2).

• The metric (2.1) could in general be Kähler or non-Kähler. However our earlier derivations from F-theory in [28] have only considered a Kähler base. Is it possible to construct a non-Kähler base from our simple F-theory derivation of [28]?

• We have not determined the warp factors F_i in our metric (2.1). Of course demanding spacetime supersymmetry will put some condition on these warp factors. Is it possible to predict the susy constraints on the metric?

• Observe that our local metric has a $z-$ fibration that is indeed constant. This is because we haven't taken the effects of the underlying seven-branes into account. In our earlier papers [26], [27] and [28] we commented that these constant fibrations will become non-constant. Can we predict this from the background equations of motion?

All the above questions require detailed analysis. So let us start with the first one. We now want to construct a fourfold whose base is a generic non-Kähler space. Later we will make this space also non-compact. We begin by considering a cubic hypersurface $Y \subset \mathbf{P}^4$ satisfying the equation

$$x_1 f + x_2 g = 0, \tag{2.6}$$

with f and g quadratic and general. There are conifolds at the four points

$$x_1 = x_2 = f = g = 0. \tag{2.7}$$

They can be resolved by blowing up the surface $S \subset Y$ defined by $x_1 = x_2 = 0$ to get a new threefold X. Blowing up a divisor doesn't change Y at its smooth points, but repairs the singularities at the conifolds.

Concretely, we introduce a variable $u = x_2/x_1$ to perform the blowup and get

$$f + ug = 0, \tag{2.8}$$

which is smooth. Over each of the conifolds, (2.8) is satisfied identically in u, so u becomes a coordinate on the $\mathbf{P}^1$. The other coordinate patch on the $\mathbf{P}^1$s is given by $v = x_1/x_2$ and proceeding similarly we complete the description of the blowup.

This X is Kähler by standard facts in algebraic geometry. Or explicitly, note that X naturally embeds as a complex submanifold of $\mathbf{P}^4 \times \mathbf{P}^1$, which is Kähler, and its Kähler metric can be restricted to X.

Now X contains four $\mathbf{P}^1$s. We can modify X by flopping only one of the $\mathbf{P}^1$s to obtain a new threefold X' with four $\mathbf{P}^1$s. Now we no longer have an embedding into $\mathbf{P}^4 \times \mathbf{P}^1$ so the previous construction of a Kähler class fails. Indeed, it can be shown that X' is not Kähler. In the last paper, we had only one conifold, and flopping it didn't destroy Kählerity since in that case we could have described the flop directly by using a different blowup. If we tried to do that here, we would end up having to flop all four $\mathbf{P}^1$s simultaneously.

Both of the resolutions X, X' are essentially Fano. More precisely, by the adjunction formula, $c_1(X) = 2H$, where H is the hyperplane class of Y pulled back to X. Similarly $c_1(X') = 2H'$, where H' is the hyperplane class of Y pulled back to X'.

So $c_1(X) \cdot C \geq 0$ for all curves C, and $c_1(X) \cdot C = 0$ only if C is one of the four $\mathbf{P}^1$s coming from the resolved conifolds. So this is very similar to the example in our last paper. The computation for X' is identical.

This short computation was intended to clarify the fact that we may no longer be restricted to a Kähler base directly in type IIB. A non-Kähler base could also lead to new gauge/gravity dualities that have not been studied before. We will comment on the possible topological string analysis for such a case in future works.

For the time being observe that the metric (2.1), which forms the base of the fourfold, cannot be regarded as the full global metric because it doesn't show the existence of three- and seven-branes. On a small patch in the neighborhood of $\mathbf{P}_0$ the metric is (2.2). It seems that there may not exist a globally defined coordinate for the system and the full global metric $-$ that takes into account the D3s, D5s, D7s and the fluxes $-$ could only be defined on patches. Nevertheless let us explore the constraints on the warp factors $F_i(\tilde{r})$ in (2.1) and then we shall restrict this on a given patch. The large $\tilde{r}$ behavior of F_i for $i = 3, 4, 5, 6$ can be expected to be

$$\begin{aligned} F_i(\tilde{r}) &= \tilde{r}^{k_i}\ G_i(\tilde{r}), \quad i = 3, 4, \\ F_j(\tilde{r}) &= a^2\ G_j(\tilde{r}), \quad j = 5, 6, \end{aligned} \tag{2.9}$$

where a^2 is the same resolution parameter that we had in (2.5). However we do not require the large $\tilde{r}$ behavior, rather the small r behavior. This can be easily arranged to be of the form

$$\begin{aligned} F_i(\tilde{r}) \ =\ & r_0^{k_i}\ G_i(r_0) + \left(k_i\ r_0^{k_i-1}\ G_i(r_0) + r_0^{k_i}\ \frac{\partial G_i}{\partial \tilde{r}}\Big|_{\tilde{r}=r_0}\right) r\ + \\ & + \frac{r_0^{k_i}}{2}\left(\frac{k_i(k_i-2)}{r_0^2}\ G_i(r_0) + \frac{2k_i}{r_0}\ \frac{\partial G_i}{\partial \tilde{r}}\Big|_{\tilde{r}=r_0} + \frac{\partial^2 G_i}{\partial \tilde{r}^2}\Big|_{\tilde{r}=r_0}\right) r^2\ + \mathcal{O}(r^3), \end{aligned} \tag{2.10}$$

where $i = 3, 4$ in general. For k_5, k_6 defined as

$$k_5 \ =\ k_6 \ =\ 2\ \log_r\ a$$

we see that for $k_3 = k_4 \equiv \kappa$ the γ defined in (2.5) can be related to G_i above only in the regime where $G_3 \approx G_4 \equiv G$. In this regime the relation that connects the space (2.1) with the one predicted by [31] is given by

$$\gamma_0'\ r_0\ -\ \kappa\ \gamma_0\ -\ r_0^{\kappa+1}\ G'(r_0)\ \to\ 0, \tag{2.11}$$

where an equality would correspond to exact identification. This is thus precisely the regime where we can trust the metric of [31].

In a generic situation when the equality in (2.11) is not maintained, we have to worry about a couple of things. One of the most important aspects is supersymmetry. As we discussed above, both the global metric (2.1) and the local version (2.2) preserve supersymmetry. However this conclusion was extracted from our F-theory picture developed in [28]. The susy model from F-theory *a priori* doesn't give any constraint on the warp factors $F_i(r)$ because the F-theory solution is written in terms of algebraic equations and not the metric. But we can use the type IIB (2,2) sigma model on this background to derive possible constraints. One of the simplest ways to start off is by using the Poisson Sigma model [52] and then include a symmetric tensor. This has already been addressed in [53], [54], [40] and the model can be written, in $(1,1)$ superspace, as

$$S = \int d^2\xi d^2\theta \big[\mathbf{\Psi}_{+\mu}\mathbf{\Psi}_{-\nu}(G^{\mu\nu} + B^{\mu\nu}) + i\mathbf{\Psi}_{(+\mu}D_{-)}\mathbf{\Phi}^\mu\big], \tag{2.12}$$

where $G_{\mu\nu}$ is the metric of (2.1) and $B_{\mu\nu}$ is the NS B-field in this background. It is easy to see that solving the equation of motion of $\mathbf{\Psi}_{\pm\mu}$, we will get

$$\mathbf{\Psi}_{\pm\mu} = iD_\pm\mathbf{\Phi}^\nu(G_{\mu\nu} - B_{\mu\nu}), \tag{2.13}$$

which when substituted back in (2.12) will give us the usual (2,2) action of [53]. One might then ask about the susy variation of $\mathbf{\Phi}^\mu$ when we are on-shell for $\mathbf{\Psi}$. The susy variation for $\mathbf{\Phi}$ is

$$\delta\mathbf{\Phi}^\mu = \epsilon^\pm\, D_\pm\mathbf{\Phi}^\nu\, J_\nu^{(\pm)\mu} - i\epsilon^\pm\mathbf{\Psi}_{\pm\rho}(G^{\nu\rho} - B^{\nu\rho})\, \mathbb{I}^\mu_\nu, \tag{2.14}$$

where $J_\nu^{(\pm)\mu}$ are two complex structures and $\mathbb{I}$ is the identity matrix. Combining (2.14) with (2.13) implies that our background would preserve (2,2) supersymmetry if we chose $\mathbf{\Phi}^\mu$ in such a way that it satisfies

$$\delta\mathbf{\Phi}^\mu = \epsilon^\pm D_\pm\mathbf{\Phi}^\nu(J_\nu^\mu \pm \mathbb{I}^\mu_\nu). \tag{2.15}$$

The first ($\theta = 0$) components of $\mathbf{\Phi}^\mu$ have the usual interpretation as complex coordinates x^μ for our space (2.1) while the components linear in θ, $D_\pm\mathbf{\Phi}\mid_{\theta=\mathbf{0}}$, are the world-sheet fermions $\lambda_\pm$. Thus, at $\theta = 0$ eq. (2.15) gives

$$\delta x^\mu = \epsilon^\pm\lambda^\nu_\pm(J \pm \mathbb{I})^\mu_\nu, \tag{2.16}$$

We could then use these components to construct *primitive* fluxes in our space but will not do so here. It is also important to notice that we have made no mention of the *choice* of the complex structures. As we know there are two allowed complex structures for our case [28], [53]. For the time being it is easy to see that there are three complex one-forms given as:

$$\begin{aligned}
\epsilon_0 &= -\sqrt{F_1}\ d\tilde{r}\ +\ i\sqrt{F_2}\ (d\tilde{\psi} + \cos\tilde{\theta}_1 d\tilde{\phi}_1 + \cos\tilde{\theta}_2 d\tilde{\phi}_2), \\
\epsilon_1 &= \sqrt{F_3}\ d\tilde{\theta}_1\ +\ i\sqrt{F_4}\ \sin\tilde{\theta}_1\ d\tilde{\phi}_1, \quad \epsilon_2 \ =\ \sqrt{F_5}\ d\tilde{\theta}_2\ +\ i\sqrt{F_6}\ \sin\tilde{\theta}_2\ d\tilde{\phi}_2.
\end{aligned} \tag{2.17}$$

These one-forms are particularly useful for constructing higher p-forms in IIB theory. What we require for our case is to allow only primitive (2,1) forms. This is possible if

$$F_3\ F_4\ -\ F_5\ F_6\ =\ 0, \tag{2.18}$$

which is motivated from the somewhat similar correspondence for the background constructed in [31]. Clearly the metric of [31] does not satisfy (2.18) and therefore breaks supersymmetry [32]. So our minimal constraint should be (2.18) on the warp factors. One easy way to impose this on (2.9) will be to consider

$$(k_3, k_4)\ =\ (4\ \log_r\ a - k_4, k_4), \qquad G_3\ G_4\ =\ G_5\ G_6, \tag{2.19}$$

where a is the resolution parameter for the resolved conifold as before. We will also have to impose the condition

$$a\ \to\ 0, \quad (G_5, G_6)\ \to\ \infty \tag{2.20}$$

such that (F_5, F_6) remain finite. In this limit the metric of [31] becomes supersymmetric which is in fact the metric of a fractional brane, and is therefore directly related to the Klebanov-Strassler metric [23]. It is also easy to see that the local metric (2.2) satisfies the primitivity constraint (2.18) and therefore preserves supersymmetry.

The way we have presented our local metric (2.2) shows only a constant dz fibration. What we need for our analysis is a metric with a non-constant $U(1)$ fibration so that it could be related to our earlier metric of [26], [27] and [28]. So the question is, under what condition does it allow non-trivial fibration? To answer this, let us first assume that we can have a generic local metric of the form

$$ds^2\ =\ dr^2 + \Big(dz + f_1(\theta_1)\ dx + f_2(\theta_2)\ dy\Big)^2 + |dz_1|^2 + |dz_2|^2, \tag{2.21}$$

where dz_1, dz_2 form the two tori (see discussions in [26], [28]).

We see the metric of (2.21) is different from (2.2) because of non-constant f_1, f_2. One immediate question that might arise is why we are getting non-constant f_i when by doing a local reduction from (2.1) we get (2.2). A very brief discussion of this was presented in [28], and here we would like to elaborate on it.

• First, observe that the metric (2.1) is not the global description for our case. A full global description will require us to have $D5s, D3s$ and seven-branes. When we ignore the $D3$ branes and keep the seven-branes far away[3] the metric resembles (2.1).

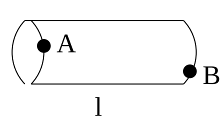
• Secondly, if we ignore the $D3$ branes and also the seven-branes, but keep only the wrapped $D5$ branes then the metric we get is the one predicted by [31]. This metric breaks supersymmetry as we discussed above (see also [32], [26], [28]).

• Thirdly, for both cases the local behaviors are almost identical except they differ in the details of the $U(1)$ fibration. So a natural question would be to ask what happens when we keep the seven-branes in the local vicinity of the wrapped $D5$ branes.

• Finally, for the metric (2.1) we should also determine the warp factors from the equations of motion. In fact the equations of motion should be used to get the full global solution that incorporates the $D3$ branes also.

Thus alternatively, in the absence of a full global solution, we could impose equation of motion constraints to determine the $f_i(\theta_i)$ in (2.21). This way we will know how much control we have on the behavior of the axion-dilaton, at least locally. Then a global solution could presumably be constructed by connecting all the local patches.

It turns out, for our case, an analytical solution can be worked out for the f_i factors using a simple set of duality maps. The map that we are interested in takes us to M-theory wherein the analysis becomes tractable. It is not too difficult to see that the f_i of (2.21) is mapped to a configuration of two points A and B on a cylinder of length l in M-theory such that these two points form a codimension 4-surface in a Calabi-Yau space. In **figure 1** below:



we have denoted the surfaces as two points A, B on an M-theory cylinder of length l with the compact angular direction related, as usual, to IIA coupling. On the other hand the

[3] To be precise, this means that the wrapped five-brane metric does get back-reacted, but the axion-dilaton are still negligible in a local neighborhood near the five-branes.

f_i also map to four-form G-fluxes given as

$$G \;=\; df_1 \oplus df_2. \tag{2.22}$$

These co-dimensional surfaces should *not* be interpreted as any kind of dynamical branes in M-theory. Right now they simply behave as localised sources of G-fluxes without having any world-volume dynamics. We will show that the only possible way they could become dynamical is if the IIB warp factors F_i in (2.1) are taken into account. So the question would be to see how the warp factors change the story.

To proceed let us first assume that the warp factors $F_i(\tilde{r})$ can be separated as products of two functions in the following way

$$F_i(\tilde{r}) \;=\; F_0^{-1}(\tilde{r})\, \mathcal{F}_i(\tilde{r}), \qquad i \;\neq\; 0 \tag{2.23}$$

A physical motivation for this conjecture comes from the F-theory origin of our metric (2.1) (see also discussion in [55] and [56]) and can be easily argued from the warped metric ansatz of [57] using the analysis of [5].

One question would now be to ask what kind of $\tilde{r}$-behavior we expect from F_0 and $\mathcal{F}_i$? We should look at the metric (2.1) for inspiration. In IIB (2.1) can be written as

$$ds^2 \;=\; F_0\, ds^2_{0123} + F_0^{-1}\, ds^2_{\mathcal{M}}, \tag{2.24}$$

where $\mathcal{M}$ can be extracted from (2.1). We see that in this form the metric fits into the D-brane ansatz, so the behavior of F_0 can be predicted as

$$F_0 \;=\; c_0 + \sum_i \frac{c_i}{\tilde{r}^{n_i}}, \tag{2.25}$$

where the n_i are some integers with c_0 being basically constant[4]. We also know that the sum over i terminates at some point so as to have a physical model and generically i cannot exceed some small number.

Once we fix a possible behavior for F_0, we can try to work out the possible $\tilde{r}$ dependence for $\mathcal{F}_i$. Better still, we can try to determine the small r behavior of $\mathcal{F}_i$. The relation

[4] The conjecture (2.25) is generic, but not generic enough for higher-dimensional branes. As we know, sometime delocalisation can change the behavior of the harmonic functions. So if we remove the restriction of positivity on n_i in (2.25) then we might capture all possible solutions. Henceforth $n_i \equiv \pm|n_i|$ unless mentioned otherwise.

between $\tilde{r}$ and r is already given in [28] (see eq. (2.13) there) and therefore we will simply use this to write the possible r behavior of the warp factors. The small r expansion is given by

$$
\begin{aligned}
\mathcal{F}_i(\tilde{r}) &= \mathcal{F}_i(r_0) \;+\; \frac{r}{\sqrt{\mathcal{F}_1(r_0)}}\,\frac{\partial \mathcal{F}_i}{\partial \tilde{r}}\Big|_{\tilde{r}=r_0} + \frac{r^2}{2\mathcal{F}_1(r_0)}\,\frac{\partial^2 \mathcal{F}_i}{\partial \tilde{r}^2}\Big|_{\tilde{r}=r_0} + \mathcal{O}(r^3) \\
&= \mathcal{F}_i(r_0) \;+\; \alpha_i r + \beta_i r^2 + \mathcal{O}(r^3),
\end{aligned} \tag{2.26}
$$

where we have kept terms up to second order in r and α_i, β_i can be easily identified.

At this point note that the local metric (2.2) is in fact an approximation where we ignore the r dependence of the warp factors $\mathcal{F}$ and keep only the constant terms $\mathcal{F}_i(r_0)$. Putting in the r dependence will give us

$$
ds^2_{\mathcal{M}} \;=\; \mathcal{A}\;dr^2 + \mathcal{B}\;(dz + f_1\;dx + f_2\;dy)^2 \;+ (\mathcal{C}\;d\theta_1^2 + \mathcal{D}\;dx^2) + (\mathcal{E}\;d\theta_2^2 + \mathcal{F}\;dy^2), \tag{2.27}
$$

with the various coefficients now defined as (we keep only up to r^2 terms)

$$
\begin{aligned}
&\mathcal{A} = 1 + \frac{\alpha_1}{\mathcal{F}_1(r_0)}r + \frac{\beta_1}{\mathcal{F}_1(r_0)}r^2; \\
&\mathcal{B} = 1 + \frac{\alpha_2}{\mathcal{F}_2(r_0)}r + \frac{\beta_2}{\mathcal{F}_2(r_0)}r^2; \\
&\mathcal{C} = 1 + \frac{\alpha_3}{\mathcal{F}_3(r_0)}r + \frac{\beta_3}{\mathcal{F}_3(r_0)}r^2; \\
&\mathcal{D} = \left(1 + \cot\,\langle\theta_1\rangle\,\sum_n\;b_n\theta_1^n\right)\left(1 + \frac{\alpha_4}{\mathcal{F}_4(r_0)}r + \frac{\beta_4}{\mathcal{F}_4(r_0)}r^2\right); \\
&\mathcal{E} = 1 + \frac{\alpha_5}{\mathcal{F}_5(r_0)}r + \frac{\beta_5}{\mathcal{F}_5(r_0)}r^2; \\
&\mathcal{F} = \left(1 + \cot\,\langle\theta_2\rangle\,\sum_n\;c_n\theta_2^n\right)\left(1 + \frac{\alpha_6}{\mathcal{F}_6(r_0)}r + \frac{\beta_6}{\mathcal{F}_6(r_0)}r^2\right),
\end{aligned} \tag{2.28}
$$

where c_n, b_n are small non-zero constants and α_i, β_i are defined as before. It is easy to see that for $(r, \theta_1, \theta_2) \to 0$ these warp factors are essentially constants, as they should be. This is consistent with our local metric ansatz. We also see that the $U(1)$ fibration is no longer required to be constant and is given in terms of $f_1(\theta_1), f_2(\theta_2)$. In fact we can write the form of the f_i also. They are given by

$$
f_1 \;=\; \sqrt{\frac{\mathcal{F}_2(r_0)}{\mathcal{F}_4(r_0)}}\left(\cot\,\langle\theta_1\rangle + \sum_n\;a_n\theta_1^n\right), \quad f_2 \;=\; \sqrt{\frac{\mathcal{F}_2(r_0)}{\mathcal{F}_6(r_0)}}\left(\cot\,\langle\theta_2\rangle + \sum_n\;d_n\theta_2^n\right), \tag{2.29}
$$

where again a_n, d_n are small constants. The above form of the f_i is perfectly consistent with (2.4). All we now need is to determine the coefficients a_n, b_n, c_n and d_n from the background equations of motion and get a closed form for the series.

This, as it stands, is a formidable task and we shall see how far we can pursue this to get the kind of answer that we want for our case. We start by considering some limits, and will show how the final answer would justify these simple assumptions. From the warp factors (2.28) we can easily construct new warp factors by introducing algebraic relations between them. For example

$$\mathcal{C}\mathcal{E} \;=\; 1 + C_0 r + C_1 r^2 + C_2 r^3 + \mathcal{O}(r^4), \tag{2.30}$$

where we have kept the r^3 term in the series assuming that C_2 coefficient is well defined. The various C_i can be easily extracted from (2.28) and are given by

$$\begin{aligned} C_0 \;&=\; \frac{\alpha_3}{\mathcal{F}_3(r_0)} + \frac{\alpha_5}{\mathcal{F}_5(r_0)}, \qquad C_2 \;=\; \frac{\alpha_3\beta_5 + \beta_3\alpha_5}{\mathcal{F}_3(r_0)\mathcal{F}_5(r_0)}, \\ C_1 \;&=\; \frac{\beta_5}{\mathcal{F}_5(r_0)} + \frac{\alpha_3\alpha_5}{\mathcal{F}_3(r_0)\mathcal{F}_5(r_0)} + \frac{\beta_3}{\mathcal{F}_3(r_0)}. \end{aligned} \tag{2.31}$$

The above is simply a redefinition: we haven't said anything yet. All we have to see are the constraints on the warp factors coming from the equations of motion. There are numerous papers that study such systems (see for example [58] and citations therein; and [59] for more recent advances) so we will not go through them in detail. The interested readers may want to see these references.

To simplify our ensuing analysis of the equations of motion we will ignore the fluxes for the time being. This will not change the expected behavior in any significant way because we will have more than one way to verify the correctness of the analysis. The background equations of motion thus put the following constraints on the various coefficients:

$$\alpha_i \;>\; \beta_i, \quad i = 3, .., 6, \qquad (b_n, c_n)\big|_{n\geq 1} \;\to\; 0 \tag{2.32}$$

with b_0 and c_0 arbitrary (but could be small); and both β_1, β_2 are not required to be smaller than α_1, α_2. In fact the equations of motion demand the following simple relations between α_i and β_i:

$$\begin{aligned} \frac{\alpha_1}{\mathcal{F}_1(r_0)} - \frac{\alpha_3}{\mathcal{F}_3(r_0)} - \frac{\alpha_5}{\mathcal{F}_5(r_0)} \;&=\; 0, \\ \frac{\beta_1}{\mathcal{F}_1(r_0)} - \frac{\beta_5}{\mathcal{F}_5(r_0)} - \frac{\beta_3}{\mathcal{F}_3(r_0)} \;&=\; \frac{\alpha_3\alpha_5}{\mathcal{F}_3(r_0)\mathcal{F}_5(r_0)}. \end{aligned} \tag{2.33}$$

These relations should get modified as higher-order terms in r are incorporated. However since we have imposed (2.32) the terms that are higher order in β_i will also get subsequently reduced. Therefore the above equations will not get corrected too much.

There are also a few more relations that do not directly appear from the equations of motion, but could be justified nevertheless. They are of the form:

$$\begin{aligned}
\frac{\alpha_3}{\mathcal{F}_3(r_0)} - \frac{\alpha_4}{\mathcal{F}_4(r_0)} &\approx 0, \qquad \frac{\beta_3}{\mathcal{F}_3(r_0)} - \frac{\beta_4}{\mathcal{F}_4(r_0)} \approx 0, \\
\frac{\alpha_5}{\mathcal{F}_5(r_0)} - \frac{\alpha_6}{\mathcal{F}_6(r_0)} &\approx 0, \qquad \frac{\beta_5}{\mathcal{F}_5(r_0)} - \frac{\beta_6}{\mathcal{F}_6(r_0)} \approx 0.
\end{aligned} \tag{2.34}$$

These equations are only approximate and their exact forms are not known as finding them requires solving higher order equations of motion. Furthermore these equations are valid only for the particular set-up of a resolved conifold or a conifold.

The relations of α_2 and β_2 with other coefficients are a little tricky to work out from equations of motion as their closed forms are difficult to derive. We have been able to work out the relations only when α_i, β_i and $\mathcal{F}_i(r_0)$ are very small. In that case there are perturbative expansions that one could use to determine the results. After the dust settles, the final results are somewhat similar to (2.33) but a little more complicated:

$$\begin{aligned}
\frac{\alpha_2}{\mathcal{F}_2(r_0)} + \frac{\alpha_4}{\mathcal{F}_4(r_0)} + \frac{\alpha_6}{\mathcal{F}_6(r_0)} &= 0, \\
\frac{\beta_2}{\mathcal{F}_2(r_0)} + \frac{\beta_6}{\mathcal{F}_6(r_0)} + \frac{\beta_4}{\mathcal{F}_4(r_0)} &= \frac{\alpha_4^2}{\mathcal{F}_4^2(r_0)} + \frac{\alpha_6^2}{\mathcal{F}_6^2(r_0)} + \frac{\alpha_4\alpha_6}{\mathcal{F}_4(r_0)\mathcal{F}_6(r_0)},
\end{aligned} \tag{2.35}$$

which could be related to (2.33) using (2.34). However observe the relative sign differences between (2.33) and (2.35). This will be crucial later.

What we now require is to evaluate the complex structures of the two tori in the metric (2.27). One can easily see that the complex structures are of the form $\tau = i\tau_2$ with vanishing real part. The imaginary part is given by

$$\begin{aligned}
\tau_2 &= 1 + \frac{1}{2}\Big(\frac{\alpha_4}{\mathcal{F}_4(r_0)} - \frac{\alpha_3}{\mathcal{F}_3(r_0)}\Big) r + \\
&\quad + \frac{1}{2}\Big(\frac{\alpha_3^2 - \beta_3\mathcal{F}_3(r_0)}{\mathcal{F}_3^2(r_0)} - \frac{\alpha_3\alpha_4}{\mathcal{F}_3(r_0)\mathcal{F}_4(r_0)} + \frac{\beta_4}{\mathcal{F}_4(r_0)}\Big) r^2 + \mathcal{O}(r^3).
\end{aligned} \tag{2.36}$$

Applying now the constraints that we derived in (2.34) the complex structure will take the final form as

$$\tau_2 = 1 + \mathcal{O}(r^3) \tag{2.37}$$

and therefore gives us square tori at least up to the order r^2. We believe this will continue to hold to arbitrary orders in r, but we haven't checked this as yet.

The readers may have already noticed that the above conclusion is perfectly consistent with our local metric ansatz that we gave in [26], [27] and [28]. In fact our present analysis should be thought of as a consistent derivation of this fact from first principles. What we now require is to evaluate the fibration structure in (2.27) to get our final form of the metric.

To do this we first apply the conditions (2.33) and (2.35) in (2.27) assuming of course the approximate constraints (2.34). The identifications are a little tedious to entangle, but one can see the following structure evolving:

$$\begin{aligned} \mathcal{A} - \mathcal{C}\cdot\mathcal{E} &= \mathcal{O}(r^3), \\ \mathcal{B} - \frac{\mathcal{D}^{-1}\cdot\mathcal{F}^{-1}}{\big(1 + b_0 \cot\langle\theta_1\rangle\big)\big(1 + c_0 \cot\langle\theta_2\rangle\big)} &= \mathcal{O}(r^3), \end{aligned} \tag{2.38}$$

which are actually evaluated under two very specific conditions: (a) the coefficients b_n, c_n for $n \geq 1$ are neglibly small, and (b) the higher order $\mathcal{O}(r^p)$ terms for $p \geq 3$ approach zero quickly.

What conditions can we impose on the constants b_0, c_0? With the weak form of the constraints (2.34), we can only say that b_0, c_0 are very small. If (2.34) is an exact equality then it is easy to show that

$$b_n = 0, \qquad c_n = 0, \qquad n \geq 0. \tag{2.39}$$

Under this condition we find some surprising simplification, with (2.38) reducing to the following condition on $\mathcal{B}$:

$$\mathcal{B} - \mathcal{C}^{-1}\cdot\mathcal{E}^{-1} = \mathcal{O}(r^3), \tag{2.40}$$

which will allow us to have some important simplification in the equations of motion for f_i, although we should remember that (2.39) is a strong condition and may not exactly hold for our background. However since the weak condition (2.34) implies that (b_n, c_n) are essentially very small, we cannot be too far off from our results.

Our next task is to figure out the relation between the warp factors $\mathcal{C}$ and $\mathcal{E}$. Observe that all other warp factors in the metric (2.27) are given in terms of either $\mathcal{C}$ or $\mathcal{E}$ or both. The relation between $\mathcal{C}$ and $\mathcal{E}$ can be easily worked out if we demand supersymmetry.

We have already seen this earlier in (2.18) and in (2.19). For our present case the susy constraint on the metric (2.27) gives us the following relation between $\mathcal{C}$ and $\mathcal{E}$:

$$\mathcal{C} ~=~ \mathcal{E}\sqrt{\frac{1+\tau_{2(2)}^2}{1+\tau_{2(1)}^2}}, \tag{2.41}$$

where $\tau_{2(i)}$ for $i = 1, 2$ are the complex structures of the two tori, evaluated earlier in (2.36) (for one of the tori). On the other hand, for a more generic conifold of the form

$$(XY)^l ~=~ (ZW)^m \tag{2.42}$$

and their resolved cases, the above equations (2.34) (and (2.41)) will pick up a relative factor of $\frac{m}{l}$ in all the relations as

$$\frac{\alpha_3}{\mathcal{F}_3(r_0)} - \frac{m}{l}\left(\frac{\alpha_5}{\mathcal{F}_5(r_0)}\right) ~\ge~ 0, \qquad {\rm for} \quad l ~\ge~ m \tag{2.43}$$

with similar coefficients for others. Observe that when $l \ne m$ then there is no equality between the coefficients. In our present analysis we will restrict ourselves to the simplest case of $l = m$ with (2.42) resolved by blowing up a $\mathbf{P}^1$.

Under the weak form of the constraint (2.34) the two complex structures are indeed equal and they both form square tori with $\tau_{(1)} = \tau_{(2)} = i$. Therefore (2.41) will simplify drastically giving rise to *one* warp factor $-$ say $\mathcal{C}(r)$ $-$ in terms of which which the metric (2.27) could be written. This implies that the final metric for our case that satisfies all the equations of motion can be written as

$$ds^2_{\mathcal{M}} ~=~ \mathcal{C}(r)^2 ~dr^2 + \mathcal{C}(r)^{-2}~\Big(dz + f_1(\theta_1)~dx + f_2(\theta_2)~dy\Big)^2 ~+\mathcal{C}(r)~|dz_1|^2 + \mathcal{C}(r)~|dz_2|^2, \tag{2.44}$$

with $\mathcal{C}(r)$ defined as before and dz_i the two tori with complex coordinates

$$dz_1 ~=~ d\theta_1 + idx, \qquad dz_2 ~=~ d\theta_2 + idy \tag{2.45}$$

We would like to remind the reader that the above metric not only satisfies all the equations of motion, but also satisfies the supersymmetry constraints. This should be contrasted with the metric of [31] which satisfies the equations of motion but not the susy constraints. We also see that the metric is exactly of the form predicted in [26], [27] and [28].

To complete the picture we have to answer the following questions now:

• How do we show that the background is Kähler?

• What are the values of the coefficients $f_i(\theta_i)$ appearing above in (2.44)?

• Can we determine the warp factors F_0 and $\mathcal{F}_i$ in the original metric (2.1)?

• We haven't introduced the effects of the seven-branes. How are the seven-branes affecting the story here?

• Finally, how is the M-theory picture that we gave earlier modified once we know the warp factors correctly?

We will start by making some comments on the Kählerity issue of the metric. Recall that in some of our earlier works [7], [8], [9], [60], [29] we have constructed non-Kähler manifolds that have a somewhat similar fibration structure as above. However the details of the fibration differ, and also there were $U(1) \times U(1)$ fibrations instead of the single $U(1)$ fibration presented here. The examples therein were compact and mostly in the heterotic theory, whereas here the manifolds are in type IIB and are non-compact. In addition to that the heterotic examples have a topology of a non-trivial $\mathbf{T}^2$ fibration over $\mathbf{K3}$ bases. Here we will have a non-trivial $\mathbf{S}^1$ fibration over a $\mathbf{T}^2 \times \mathbf{T}^2$ base.

A further analysis on the issue of Kählerity can only come after we have evaluated all the unknown coefficients in the metric (2.44). A formal analysis of the equations connecting the fibration coefficients f_i with the warp factors will give us a simple result where $a_1 = d_1 = 1$ and $a_n = d_n = 0$ for $n \neq 1$ in (2.29). But we can do a little better than that. As we pointed out earlier in (2.22), the f_i map to four-form G-fluxes in M-theory. We will presume that these fluxes can be globally defined because then they would consistently couple with the points **A** and **B** discussed in fig. 1 above, to form sources. This would in turn mean that the f_i now satisfy the standard source equations *globally*. The resulting analysis turns out to be straightforward, but long and tedious. Interested readers may want to see related details in [61], [62], [63], [59] etc. The final results are two decoupled equations connecting the various coefficients with the warp factor $\mathcal{C}$ as:

$$\begin{aligned} \frac{\partial f_1}{\partial \theta_1} - \frac{\partial \mathcal{C}}{\partial r} + f_1 \cot\theta_1 &= 0; \\ \frac{\partial f_2}{\partial \theta_2} - \frac{\partial \mathcal{C}}{\partial r} + f_2 \cot\theta_2 &= 0. \end{aligned} \tag{2.46}$$

To solve the above equations we will assume that the radial coordinate is small. This is justifiable because we are in the local regime where r is indeed small. Secondly we will take $\alpha_3 >> \beta_3$. To justify this we need to go back to (2.32) where we argued the weaker form $\alpha_3 > \beta_3$. However since $\mathcal{O}(r^2)$ terms are small, this could be justified and hence $\mathcal{C}$ will become simpler:

$$\mathcal{C} \ = \ 1 + \left(\frac{1}{\mathcal{F}_3(r_0)\sqrt{\mathcal{F}_1(r_0)}} \frac{\partial \mathcal{F}_3}{\partial r}\Big|_{r=r_0}\right) r \ \equiv \ 1 + Q\, r, \tag{2.47}$$

where we have written the partial derivative w.r.t. r instead of $\tilde{r}$ to avoid clutter, and Q is defined accordingly. Thus plugging (2.47) into the two equations (2.46) we get our two fibration coefficients f_i for $\theta_i \neq 0$ as

$$f_1(\theta_1) \ = \ Q \cot\theta_1, \qquad f_2(\theta_2) \ = \ Q \cot\theta_2, \tag{2.48}$$

which is *exactly* what we had predicted earlier in [26], [27] and [28][5]! The final metric therefore takes the following form:

$$\begin{aligned} ds^2_{\mathcal{M}} \ &= \mathcal{C}(r)^2\, dr^2 + \mathcal{C}(r)^{-2}\left(dz + Q \cot\theta_1\, dx + Q\cot\theta_2\, dy\right)^2 + \\ &\qquad + \mathcal{C}(r)\,(d\theta_1^2 + dx^2) + \mathcal{C}(r)\,(d\theta_2^2 + dy^2), \end{aligned} \tag{2.49}$$

which is the same as our predicted local metric in the limit where $r \to 0$ and $\mathcal{C} \to 1$. Thus we have justified all the choices made in understanding the duality cycle for geometric transition.

Now before we go back to the issue of Kählerity of our metric, let us ask how the wrapped five-branes and the seven-branes show up in our metric (2.49). We have already seen how to put five-branes in our setup (see sec. 3 of [26]). In our present formulation, the existence of five-branes would be signalled by the warp factor $F_0(r)$ in (2.23). In fact singularities of F_0 will be related to the presence of localised five-brane charges. For the local region, where we are located near $\tilde{r} = r_0$ and have access only to the small neighborhood governed by the coordinates (r, x, y, θ_i, z) we will not detect the singularity and F_0 will be essentially constant.

Clearly then the global behavior with wrapped D5 branes has to incorporate these issues. The seven-branes on the other hand, not only introduce the singularities (associated with the position of the seven branes) but also change the *topology* of the underlying space by converting the tori to spheres [64]. For example if the F-theory seven-branes are kept at a point on the (x, θ_1) torus, then a large number of such seven-branes will compactify that direction to an approximate spherical topology. So our original metric (2.1) with $\mathcal{F}_i$ given as (2.26) is unlikely to capture the global picture.

[5] At least up to the coefficient Q which, since it is a constant, can be absorbed in the definitions of dx, dy.

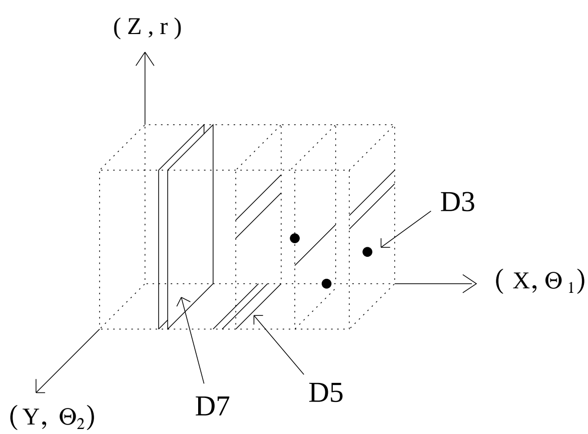
One might also wonder about the case when we introduce back the D3 branes along with the D5s and the seven-branes. To solve for the metric in the presence of all these branes is quite formidable, and at present no known solutions exist. Some attempts to address this issue with D3s and D7s have been discussed earlier in [65] based on a F-theory picture advocated in [66]. Thus we cannot go too far in r in the original metric (2.1) without hitting an F_i singularity, and we cannot go too far in the angular direction without encountering a possible topology change. Thus choosing a local patch like (2.49) from (2.1) seems like the only known solution for the system.

Let us analyse this a bit more. From [28] we know that the F-theory torus can degenerate over a 2d surface given by $dz_1 = dx + id\theta_1$. In the presence of all the branes, the generic metric along the z_1 direction is given by

$$ds_1^2 \ = \ |f(r, z_1, \bar{z}_1) \ dz_1|^2 \ = \ \mathcal{C}(r)\left|e^{\frac{\phi}{2}}dz_1\right|^2 \quad \longrightarrow \quad \mathcal{C}(r)|dz_1|^2, \tag{2.50}$$

when the dilaton $\phi \to 0$. Thus moving the seven branes far away in the z_1 space will imply that the warp factors are given simply by $\mathcal{C}(r)$ and have no angular dependence. This is again consistent with our earlier choice in [26], [27] and [28].

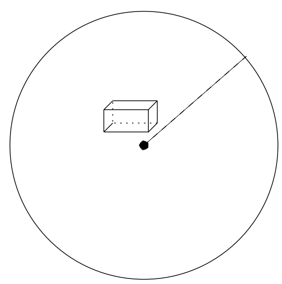
Now that we know the precise local metric and also the effects of moving the seven branes, we should re-analyse our configuration. In **figure 2** below:



the local patch in six-dimensional space is denoted by a cubical space. We denote the (x, θ_1) direction as the x-axis, the (y, θ_2) direction as the y-axis, and the (z, r) direction as the z-axis. This way the full six-dimensional space can be represented. In this space the seven-branes are two dimensional surfaces that partition the patch into two regions.

Clearly the seven-branes are stretched along the (y, θ_2, z, r) directions and are points in the (x, θ_1) direction. The D5 branes wrap the (y, θ_2) direction and appear as 1D lines inside the cube. It is easy to see that the D5 branes are parallel to the seven-branes and can be moved *away* from them along the (x, θ_1) direction. The D3 branes (which wouldn't be present if we wanted to study only the IR of gauge theory, but would be there in the full UV story) appear as points inside the cubical space.

In the figure above it is easy to see that there could be strings stretched between the seven branes and the five-branes and also between the D3 branes. These strings that stretch between the seven-branes and the D3s and D5s give rise to fundamental multiplets in $\mathcal{N} = 1$ gauge theory. Similarly the strings stretched between the D5s and D3s give rise to bi-fundamental multiplets. The fundamental multiplets can be made very massive by moving the seven-branes away, as can be easily seen by embedding this patch in the full global geometry. In **figure 3** below:



our local patch is shown inside the full geometry. We have also kept the patch near the origin to emphasis the IR behavior of our configuration. The seven-branes and the D3 branes can be moved out of the patch to construct pure $\mathcal{N} = 1$ gauge theory. The origin of the space will have the topology of a resolved conifold.

What does all this say about the supersymmetry of our model? As long as our local metric is of the form (2.49) with $\mathcal{C} \to 1$ we would preserve supersymmetry with both five-branes and seven-branes. From an F-theory point of view, the fourfold could be constructed with a T^2 fiber degenerating along the (x, θ_1) direction in the local geometry, as shown in [28]. This way we would know the full global topology but not the global metric. Only the local metric is known so far. The metric (2.1), although written in terms of global coordinates, cannot give us the global metric because the metric has no information about the three-branes and seven-branes. Once we get the local patch (2.49) we can insert the other branes and combine all the cubical patches (see figure above) to get the full global picture.

Furthermore, from the figure above, we see that there is also an interesting regime where we can have light fundamental multiplets in the IR. This will be the case when the five-branes are near the seven-branes. In fact the five-branes could possibly dissolve in the seven-branes as first Chern class of gauge bundles. For such a thing to happen the topology of the wrapped seven-branes is important. In the local geometry the seven-branes wrap the torus (y, θ_2) along with the (r, z) direction. The local geometry along the (r, z) direction at a constant value of (x, y) is given by

$$ds^2_{rz} \ = \ (1 + Qr)^2 \ dr^2 \ + \ \frac{dz^2}{(1 + Qr)^2} \ = \ dR^2 \ + \ \frac{dz^2}{1 + 2QR}, \tag{2.51}$$

where R and r are related in the standard way: $R = r + \frac{Qr^2}{2}$. Both R and r are local variables, and so the z circle would decrease as we move away from the origin. This behavior is only local of course and in the absence of a global metric it is difficult to predict the behavior of the (r, z) metric. In case, however, the z behavior continues to persist globally, then topologically the (r, z) metric will become the metric of a squashed sphere, and then the D5 brane charges Q_5 will be given by

$$Q_5 \ = \ \int_{\mathbf{P}^1} \mathrm{tr} \ F \ \equiv \ 2\pi c_1(F), \tag{2.52}$$

which is the first Chern class of the vector bundles on the seven-branes. The trace is over the adjoint representation of a subgroup of the full global group (which could be as big as $E_8 \times E_8$).

The possibility of the existence of a bound state in our system may give us a possible hint of the existence of an *obviously* supersymmetric configuration that has a close resemblance to our present setup. Imagine we start with a F-theory configuration with only seven-branes and no five-branes. We could then isolate *one* seven-brane out of the full bunch and put fluxes on it *a là* (2.52) to create the five-branes. This is a supersymmetric configuration [67] as the strings between five-branes and the seven-brane become tachyonic resulting in a negative energy that effectively reduces the total energy of the system to its bound state energy. The other strings that connect the five-branes with the rest of the seven-branes are naturally massive (for details on this see the second reference of [67]). This would also mean, going back to the solution of [31] and taking a metric like [31] with additional seven-branes, that we can *soak* the five-branes on an isolated $D7$ brane to form a bound state preserving supersymmetry. Since most of the seven-branes are far away from the bound system, the axion-dilaton in the local neighborhood of the $D5$ branes is negligible. Thus locally the metric would resemble the solution of [31] but the global picture would be different, as one might expect. In this way we can resolve all the issues in our model which would be difficult to study in the models presented in the literature.

So far we have been ignoring the fluxes. It is time now to take them into account. Of course choosing fluxes that satisfy equations of motion will not suffice. We need fluxes that also preserve supersymmetry. A generic choice of fluxes *at* and *away* from the orientifold point is given earlier in [28]. Here, for simplicity, we shall consider only a simple choice. For B_{NS} and B_{RR} we choose

$$\begin{aligned} B_{NS} &= \mathcal{B}_{y\theta_1}(\theta_2), \qquad B_{RR} = \tilde{\mathcal{B}}_{xz}(r), \\ H_{NS} &= \mathcal{H}_{y\theta_1\theta_2} = \partial_{[\theta_2}\mathcal{B}_{y\theta_1]}, \quad H_{RR} = \tilde{\mathcal{H}}_{xzr} = \partial_{[r}\tilde{\mathcal{B}}_{xz]}, \end{aligned} \tag{2.53}$$

where the complete anti-symmetrisation implies the situation when we are away from the orientifold point. It is easy to see that the supersymmetry condition on fluxes,

$$\tilde{\mathcal{H}}_{xzr} = *_6 \, \mathcal{H}_{y\theta_1\theta_2}, \tag{2.54}$$

can be preserved with $*_6$ being the Hodge dual in our six-dimensional local metric. Both the NS and the RR fluxes survive the orientifold projection: $\Omega \cdot (-1)^{F_L} \cdot \mathcal{I}_{x\theta_1}$ but could be defined away from the orientifold point also. Finally the seven-form field strength on the wrapped D5 branes is given by

$$\mathcal{H}_7 = \partial_{[\theta_1}\mathcal{C}_{0123y\theta_2]} = *_{10} \, \tilde{\mathcal{H}}_{xzr} \tag{2.55}$$

and therefore gives the five-form sources $\mathcal{C}_{0123y\theta_2}$ on the wrapped D5 branes with $*_{10}$ being the Hodge dual in the full ten-dimensional space.

Before moving ahead we would like to make the following observation: from the orientation of the B_{NS} field (2.53), we see that one component of the B_{NS} field is along the direction of the five-branes, whereas the other component is orthogonal to it. With the existence of the field strength $\mathcal{H}_{y\theta_1\theta_2}$ we are guaranteed that the B_{NS} field cannot be gauged away, and therefore will give rise to *dipole* deformations of $\mathcal{N} = 1$ gauge theory [34]! This has recently been addressed as β-deformation of gauge theories in [35]. One of the immediate advantages of this is to decouple KK modes. We will discuss this more later in the paper.

Coming back to our earlier discussion of the cylindrical configuration in M-theory, we can now quantify the flux (2.22). These G-fluxes are in principle *different* from the F-theory G-fluxes with components

$$G \;=\; g_1(\theta_2)\; d\theta_1 \wedge d\theta_2 \wedge dy \wedge dx^a \;+\; g_2(r)\; dx \wedge dz \wedge dr \wedge dx^3, \tag{2.56}$$

where x^a denotes the eleventh direction and g_1, g_2 are sufficiently different from f_1, f_2, but their values are not yet determined. Existence of two different G-fluxes for the same type IIB picture implies that there could exist two different *dual* configurations. Both pictures may cover varying amounts of information for the type IIB set-up. These two dual configurations should *not* be confused with the *gravity* dual already present in type IIB! Existence of so many dual configurations also means that we might be able to interpolate between them.

But there is more to that. There are *three* different dual configurations that are related to the $\mathcal{N} = 1$ gauge theory: M-theory with an MQCD-like brane configuration [56], M-theory compactification on a G_2 manifold [68], and an F-theory compactification on a fourfold [55]. The configuration with G-fluxes (2.56) is defined exclusively on an elliptically fibered fourfold [28], [26], whereas the other one with G-fluxes (2.22) can interpolate between the two descriptions [56] and [68]. In the limit where the description is completely in terms of branes *a là* [56], one can show that the weakly coupled type IIA description is when the co-dimension four surfaces in CY, A and B, are connected by a D-brane or an anti D-brane (see fig. 1 above). We can go from one picture to another by simply crossing A and B i.e. $l \;\to\; -l$. In general A and B can move along the x^a circle as well in M-theory. In the limit when the co-dimension four surfaces behave as NS5 branes, we know that for weakly coupled type IIA, $l = 0$ is a susy preserving fixed point where A and B are on top of each other [56].

The third dual background of M-theory on a G_2 structure manifold is by now well known from the supergravity solution presented in [26], [27], [28] that gives the original conjecture of [48] a firmer footing.

3. The heterotic background: Torsion and Non-Kähler manifolds

In [27] we showed how we can construct possible gauge/gravity dual models in the heterotic and type I $SO(32)$ theories. The construction relied on identifying a possible orientifold corner of type IIB theory and then U-dualising the picture to go to the heterotic side. Following duality chains we were able to give a *local* description of the gravity dual of wrapped heterotic NS5 branes in [27]. In [28] we realised that there is a possible way to construct the full global picture of the gravity dual. In fact we were able to construct the explicit metric using some special identifications, and found that under some simplifying assumptions on the warp factors the global metric looks very similar to the Maldacena-Nunez type metric [24] in the IR and to the [69] type metric in the UV. For a more generic choice of the warp factors the metric may not resemble any known configuration, so would give rise to a new class of supergravity solutions with a well defined UV and IR behavior.

In [28], as discussed above, we found the UV description completely. This is of course the metric *after* geometric transition (in the language of [26]). Here we would like to address the issue of global completion for the metric *before* geometric transition. But before that let us clarify a few subtle points.

• The heterotic background that we proposed in [27] and [28] is *not* U-dual to the type IIB background that we discussed above. The orientifolding effect used in type IIB is very different from the one used to go to the U-dual heterotic background. The existence of *two* different orientifolding possibilities for the same type IIB background reflects the differences between the existence of isometries and the existence of invariances. What we saw in [27] was that the isometry directions do not always imply metric invariances. In fact even in the local limit the metric was not invariant under orientifolding of two isometry directions. This is where our background differs from the one studied earlier in [5], [7], [8], [9]. In those works, invariance and isometry went hand in hand, and therefore orientifolding was easy. Here an *obvious* orientifolding *does not* lead us to the heterotic background. The standard orientifolding gives us a supersymmetric background with seven branes that we were able to use effectively to study $\mathcal{N} = 1$ gauge theories. This orientifold operation may be related to the orientifold of T-dual brane configuration proposed in [70]. The orientifolding that we used to go the the heterotic side in [27], [28] keeps only part of the original metric invariant.

• Our next venture was to identify the invariant part of the metric that should also solve the background equations of motion, along with the susy conditions. Analysing the superpotential showed that the invariant part is in fact the *trivial* $U(1)$ fibration over the $\mathbf{T}^2 \times \mathbf{T}^2$ base inside our local geometry (2.49). Recall that the original type IIB theory has the topology of a non-trivial $U(1)$ fibration over a $\mathbf{T}^2 \times \mathbf{T}^2$ base locally. The type IIB tori (x, θ_1) and (y, θ_2) start off as square tori, but then are boosted to provide the correct mirror backgrounds. On the other hand, the two base tori in the heterotic side are succinctly constructed as (x, y) and (θ_1, θ_2) with possible non-trivial complex structures between them (see the discussion in [27]). Minimising the type IIB superpotential also hints that the heterotic tori could be more general than square tori to preserve supersymmetry.

• The semi-toroidal geometry $\mathbf{T}^2 \times \mathbf{T}^2 \times \mathbf{S}^1 \times \mathbb{R}^+$ with square tori unsurprisingly turns out not to be the most generic solution. This is of course consistent with earlier studies on flux compactifications [5], [7], [20]. A particular choice of fluxes may lead to tori with non-trivial complex structures. In fact there is already a complex structure inherited from the parent metric (2.49)

$$\tau_1 \;=\; \frac{1}{2}\left(\sin\, 2\langle\theta_1\rangle\, \cot\, \langle\theta_2\rangle + i\, \frac{2\, \sin^2\langle\theta_1\rangle}{\sqrt{\langle\alpha\rangle}}\right), \qquad \tau_2 \;\approx\; i, \tag{3.1}$$

which should be allowed for specific choices of the background B_{NS} fields $(b_{x\theta_1}, b_{y\theta_2})$ in the notation of [27]. The constant $\langle\alpha\rangle$ is defined in [26]. On the other hand, after geometric transition the parent metric allows the following complex structure to be inherited by the two tori:

$$\tau_1 \;\approx\; i, \qquad \tau_2 \;=\; a_1\, \cot\, \langle\theta_1\rangle\, \cot\, \langle\theta_2\rangle + i\sqrt{a_2 - a_3\, \cot^2\, \langle\theta_1\rangle\, \cot^2\, \langle\theta_2\rangle}, \tag{3.2}$$

where the a_i are some constants that could be determined from the duality cycle of [26] and [27]. Thus we see that the semi-toroidal geometry has some inherent complex structures already for the two $\mathbf{T}^2$.

• The inherited complex structure before geometric transition dualises in the heterotic theory to a more complicated fibration of the first tori. In fact this non-trivial fibration is the key reason why the heterotic metric fails to retain Kählerity [27]. The local analysis of the system was presented in [27]. So here we will first explore a convenient representation of the local metric that explicitly shows the wrapped brane configuration (or its possible generalisation) and later see how far we can go to elucidate the full global picture in a somewhat similar vein as we explored the global metric after geometric transition in [28]. Our starting point is the U-dual metric of [27] that is an explicit solution of the heterotic equations of motion. The proposed local metric for our space is worked out in [27] and will be given as

$$\begin{aligned} ds^2 \;&= d_1\, (dy - b_{yj}\, d\zeta^j)^2 + d_2\, (dx - b_{xi}\, d\zeta^i)^2 + d_3\, dr^2\, + \\ &\quad - 2d_4\, (dx - b_{xi}\, d\zeta^i)(dy - b_{yi}\, d\zeta^j) + d_5\, dz^2 + d_6|d\chi_2|^2, \end{aligned} \tag{3.3}$$

which is similar to the type I metric that we had in [27], as it should be. The various coefficients appearing in (3.3) are defined as follows:

$$\begin{aligned} d_1 &= \sqrt{\langle\alpha\rangle}\big(1+\cot^2\,\langle\theta_1\rangle\big), \\ d_2 &= \sqrt{\langle\alpha\rangle}\big(1+\cot^2\,\langle\theta_2\rangle\big), \\ d_3 &= \frac{\gamma'(r_0)\sqrt{H(r_0)}}{\sqrt{\langle\alpha\rangle}}, \qquad d_5 = \frac{1}{\sqrt{\langle\alpha\rangle}}, \\ d_4 &= \sqrt{\langle\alpha\rangle}\,\cot\,\langle\theta_1\rangle\,\cot\,\langle\theta_2\rangle. \end{aligned} \tag{3.4}$$

The metric has the usual fibration along the dx and the dy directions and the base is given by the $(\theta_1, \theta_2, r, z)$ coordinates. The background B field (whose components may depend on r, θ_1 and θ_2 only) can be written in terms of the U-dual IIB B-fields with legs along (x, θ_1) and (y, θ_2) directions. These B-fields – that will serve as the torsion – and the coupling constant are given by

$$\begin{aligned} H^{\rm het} \equiv H &= \mathcal{H}^b_{xz\theta_1}\,dy\wedge dz\wedge d\theta_2 - \mathcal{H}^b_{yz\theta_2}\,dx\wedge dz\wedge d\theta_1 + \\ &+ \mathcal{H}^b_{xzr}\,dy\wedge dz\wedge dr - \mathcal{H}^b_{yzr}\,dx\wedge dz\wedge dr; \\ g^{\rm het} &= \frac{1}{\sqrt{\langle\alpha\rangle}}. \end{aligned} \tag{3.5}$$

We make the following observations:

• Along with the solution to the DUY equation, (3.3) and (3.5) will specify the complete background.

• The fibrations in (3.3) are due to $b_{x\theta_1}, b_{y\theta_2}$ which do survive the orientifold action for this case[6].

• The B-fields that we took in [28] to study the global geometry after geometric transition in the heterotic theory are along $(b_{x\theta_2}, b_{y\theta_1})$.

• As expected the local metric has only constant coefficients. This is consistent with the fact that the dilaton (3.5) is a constant (see [71], [72], [5], [7] for details).

From all the above discussion we see that we can keep all the components of the B-fields: $(b_{x\theta_1}, b_{x\theta_2}, b_{y\theta_1}, b_{y\theta_2})$ and study the deformation of the metric. It is a straightforward

[6] Recall that the orbifold actions that led to (3.3) and (3.5) are along the (x, y) directions. The orbifold actions in sec. 2.1 are along (x, θ_1) that will not allow these components of B-fields.

exercise to work out the deformation s_i, and the final metric with deformations is given by:

$$G = \begin{pmatrix} G_{xx} & G_{xy} & G_{xz} & G_{x\theta_1} & G_{x\theta_2} \\ G_{xy} & G_{yy} & G_{yz} & G_{y\theta_1} & G_{y\theta_2} \\ G_{xz} & G_{yz} & G_{zz} & G_{z\theta_1} & G_{z\theta_2} \\ G_{x\theta_1} & G_{y\theta_1} & G_{z\theta_1} & G_{\theta_1\theta_1} & G_{\theta_1\theta_2} \\ G_{x\theta_2} & G_{y\theta_2} & G_{z\theta_2} & G_{\theta_1\theta_2} & G_{\theta_2\theta_2} \end{pmatrix}$$

$$= \begin{pmatrix} d_2 & -d_4 & 0 & s_1 - d_2\ b_{x\theta_1} & s_2 + d_4\ b_{y\theta_2} \\ -d_4 & d_1 & 0 & s_3 + d_4\ b_{x\theta_1} & s_4 - d_1\ b_{y\theta_2} \\ 0 & 0 & d_5 & 0 & 0 \\ s_1 - d_2\ b_{x\theta_1} & s_3 + d_4\ b_{x\theta_1} & 0 & d_6 + \mathcal{A}_1 & -d_4\ b_{x\theta_1} b_{y\theta_2} - d_4^{-1} s_i s_j \\ s_2 + d_4\ b_{y\theta_2} & s_4 - d_1\ b_{y\theta_2} & 0 & -d_4\ b_{x\theta_1} b_{y\theta_2} - d_4^{-1} s_i s_j & d_6 + \mathcal{A}_2 \end{pmatrix} \tag{3.6}$$

where we have already defined the d_i in (3.4), and the various deformations in the metric are now given as

$$\begin{aligned} s_1 &= d_4\ b_{y\theta_1}, \quad s_2 = -d_2\ b_{x\theta_2}, \quad s_3 = -d_1\ b_{y\theta_1}, \quad s_4 = d_4\ b_{x\theta_2}, \\ \mathcal{A}_1 &= d_1\ b_{y\theta_1}^2 + d_2\ b_{x\theta_1}^2 - 2d_4\ b_{y\theta_1}\ b_{x\theta_1}, \quad \mathcal{A}_2 = d_1\ b_{y\theta_1}^2 + d_2\ b_{x\theta_2}^2 - 2d_4\ b_{y\theta_2}\ b_{x\theta_2}, \end{aligned} \tag{3.7}$$

along with the following definition: $s_i s_j = s_1 s_4 - s_2 s_4 - s_1 s_3$. The above construction will be the most generic metric that we can study with constant background dilaton. It is also easy to see that if the χ_2 torus had a nontrivial complex structure i.e Re $\tau_2 \neq 0$, then the only changes we would need to incorporate are

$$\begin{aligned} G_{\theta_1\theta_2} &\to G_{\theta_1\theta_2} + d_6 \text{ Re } \tau_2; \\ G_{\theta_2\theta_2} &\to G_{\theta_2\theta_2} - d_6\big(1 - |\tau_2|^2\big). \end{aligned} \tag{3.8}$$

It is also clear that the B-field in (3.5) will now have to change to incorporate other components. This is easy to work out, so we shall not do it here. Instead we want to concentrate on various interesting cases that can be studied from the above metric (3.6) by going to different allowed limits.

3.1. Heterotic NS5-branes wrapped on a two-cycle of a torsional manifold

This is the situation where we switch on all the components of type IIB B_{NS}-fields. In terms of our orientifold construction this would be the case when we are away from the orientifold point. The d_i coefficients of the metric (3.6) are constants at least in the local limit and therefore could be fixed at some values by coordinate redefinition and scalings of B fields. In fact what we require is to have

$$d_1 \; = \; d_4\left(\frac{b_{x\theta_1}}{b_{y\theta_1}}\right), \qquad d_2 \; = \; d_4\left(\frac{b_{y\theta_1}}{b_{x\theta_1}}\right), \tag{3.9}$$

leaving us a metric that is independent of coordinate reversal $\zeta_i \;\to\; -\zeta_i$ where ζ_i are the local coordinates, as well as invariant under type IIB *orbifold* action $(x,y) \;\to\; (-x,-y)$. However this parity transformation makes sense only if the matrix

$$\Theta \; = \; \begin{pmatrix} b_{x\theta_1} & b_{x\theta_2} \\ b_{y\theta_1} & b_{y\theta_2} \end{pmatrix} \tag{3.10}$$

has a vanishing determinant $\det\Theta \; = \; 0$.

Our aim in allowing a parity invariant metric is to convert our background heterotic metric (3.6) to the following form:

$$ds^2 \; = \; |dz_1|^2 + |dz_2|^2 + d\tilde{z}^2 + dr^2, \tag{3.11}$$

where $dz_1 = d\tilde{x} + \tau_3\; d\tilde{y}$ and $dz_2 = d\tilde{\theta}_1 + \tau_4\; d\tilde{\theta}_2$. The tilde-coordinates are defined as

$$d\tilde{x} \; = \; \sqrt{d_2}\; dx, \qquad d\tilde{y} \; = \; \sqrt{d_2}\; dy, \qquad d\tilde{\theta}_i \; = \; \sqrt{d_6}\; d\theta_i, \tag{3.12}$$

where d_i are defined in (3.4). The complex structures of the two tori are now

$$\tau_3 \; = \; \frac{1}{2\sqrt{d_2}}\Big(- d_4 + i\sqrt{4d_1d_2 - d_4^2}\Big), \qquad \tau_4 \; = \; \tau_2 \tag{3.13}$$

with all the coefficients defined earlier in (3.7). Observe that for the dz_2 torus, we get back the original complex structure.

The metric (3.11) is not quite the metric that we might expect for the wrapped five branes. This is because the coefficients of the metric are strictly constant. To see what could possibly change when we make the coefficients non-constant, we have to follow a series of steps that would take us to type IIB and back *not* as an orientifold operation but through sigma-model identification. This should be reminiscent of what we did for the heterotic case in [28] (see sec. 3.1 and 3.2 therein). First, however, let us change our definition of dz. We want to define

$$d\tilde{z} \;=\; dz + a \,\cot\, \langle\theta_1\rangle\, d\tilde{x} + b\, \cot\, \langle\theta_2\rangle\, d\tilde{y}, \tag{3.14}$$

which, being a total derivative, doesn't change anything in the original metric (3.11). The metric (3.11) can be rewritten as

$$ds^2 \;=\; dr^2 + \left(dz + a\, \cot\, \langle\theta_1\rangle\, d\tilde{x} + b\, \cot\, \langle\theta_2\rangle\, d\tilde{y}\right)^2 + |dz_1|^2 + |dz_2|^2, \tag{3.15}$$

which looks very close to the local metric (2.2) except (a) we now have non-trivial complex structures on the two tori, and (b) our tori are (x, y) and (θ_1, θ_2) whereas in (2.2) the tori are (x, θ_1) and (y, θ_2). Of course both in (3.15) and (2.2) the definitions of base tori are not crucial, so a formal identification of the metrics will help us to rewrite (3.15) in such a way as to reflect the wrapped brane metric (at least locally).

In sec. 2.1 we saw how we could go from (2.2) to (2.49) by solving the equations of motion with a warped metric ansatz. Now the question is, can we follow the same argument for (3.15) also? This is where sigma-model identification of heterotic and type IIB helps. More specifically, the steps that we would like to follow are:

• Define the heterotic sigma model on this background with metric (3.11) along with torsion and gauge bundle.

• Use the sigma model identification by redefining vector bundles with torsional connection *only* in the left-moving sector. The right moving sector remains unchanged.

• Absence of anomalies and Chern-Simons corrections tells us that the resulting sigma model should be viewed as the sigma model in type IIB theory on the same metric (3.11).

• The heterotic vector bundle will now have one-to-one correspondence with *torsional* curvature in type IIB theory.

• The type IIB metric (3.11) can then be manipulated, as before, to get the final metric of the form (2.49).

• The metric (2.49) could then be transferred back to the heterotic side by performing the reverse transformations from type IIB to the heterotic side.

The above set of steps generically give us a non-Kähler manifold in type IIB with torsion, instead of a conformally Kähler. The background preserves minimal supersymmetry. That this is not a contradiction can be easily shown: because of the existence of an underlying type IIB U-duality symmetry a supersymmetric non-conformally Kähler manifold can exist in the presence of torsion and zero RR three-forms (see [28] for details).

To implement the above set of steps, we need the sigma model of heterotic theory on the background (3.11). We will follow the notations of [72], where the left and right moving fermions are called S^ρ and Ψ^A respectively. The worldsheet interactions of these fermions with the background gauge fields are given by the following terms of the lagrangian:

$$S_{\rm interaction} = \frac{i}{8\pi\alpha'}\int\Big(S^\rho\omega^{ab}\sigma_{ab}^{\rho\sigma}S^\sigma - iF_{ij(AB)}\sigma_{\rho\sigma}^{ij}S^\rho S^\sigma\Psi^A\Psi^B + \Psi^A A_i^{(AB)}\bar\partial X^i\Psi^B\Big), \tag{3.16}$$

where by definition $F_{ij(AB)} = F_{ij}^a M_{AB}^a$ and the index a labels the adjoint of the gauge group; and along with (3.16) there is the standard kinetic term

$$S_{\rm kinetic} = \frac{1}{8\pi\alpha'}\int\Big(\partial X\bar\partial X\cdot(\mathbf{g}+\mathbf{B}) + iS\cdot\partial S + i\Psi\cdot\bar\partial\Psi\Big), \tag{3.17}$$

where indices are contracted accordingly. The total action is of course the sum of the two actions (3.17) and (3.16). The supersymmetry of the sigma model at this stage is just $(0,1)$ and *not* $(0,2)$ as one might have naively expected. If we now employ the following identification:

$$A_i^{AB} = \begin{pmatrix}\omega_i^{ab} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}, \qquad \Psi^A = \begin{pmatrix}S^{\dot q}\\ \Psi^9\\ \dots\\ \dots\\ \Psi^{32}\end{pmatrix}, \tag{3.18}$$

then it is easy to show that the interaction term (3.16) changes to

$$\tilde S_{\rm interaction} = \frac{i}{8\pi\alpha'}\int\left(S^\rho\omega^{ab}\sigma_{ab}^{\rho\sigma}S^\sigma + S^{\dot\rho}\omega^{ab}\sigma_{ab}^{\dot\rho\dot\sigma}S^{\dot\sigma} - \frac{i}{2}\mathcal{R}_{ijkl}\sigma_{\dot\rho\dot\sigma}^{ij}\sigma_{\kappa\gamma}^{kl}S^{\dot\rho}S^{\dot\sigma}S^\kappa S^\gamma\right). \tag{3.19}$$

On the other hand the kinetic term (3.17) remains more or less unchanged. The only change therein is

$$\Psi\cdot\bar\partial\Psi \;\to\; S^{\dot\sigma}\,\bar\partial S^{\dot\sigma} + \sum_{A=9}^{32}\,\Psi^A\bar\partial\Psi^A, \tag{3.20}$$

where $\Psi^9, ...\Psi^{32}$ are completely decoupled because of our choice of A_i in (3.18). Now defining $\tilde S \equiv \tilde S_{\rm kinetic} + \tilde S_{\rm interactions}$ and $S_{\rm het} \equiv S_{\rm kinetic} + S_{\rm interactions}$ we see that

$$S_{\rm het} \;\to\; \tilde S \;=\; S_{\rm IIB} \;+\; \frac{i}{8\pi\alpha'}\sum_{A=9}^{32}\,\Psi^A\bar\partial\Psi^A, \tag{3.21}$$

where $S_{\rm IIB}$ is the type IIB worldsheet lagrangian. Thus up to the decoupled Ψ^A fermions, the heterotic background maps to the type IIB background.

This is a very interesting situation now. The heterotic metric (3.15) is now the metric in type IIB also. Thus all the manipulations that we performed in type IIB (in sec. 2.1) should apply to this metric also. In particular the final metric that we will get in type IIB will *not* be Kähler. In fact there is no reason for the metric to be even conformally Kähler. Furthermore we expect the metric of the two-cycle on which we have wrapped NS5 branes to be of the form

$$ds^2_{2-{\rm cycle}} ~ = ~ \Big(|\tau_3|^2 + a_o\Big)~d\tilde y^2 ~ + ~ \Big(|\tau_4|^2 + b_o\Big)~d\tilde\theta_2^2, \tag{3.22}$$

where $\tau_{3,4}$ have been defined earlier in (3.13) and a_o, b_o are the possible corrections (to be determined below).

In the limit $\tau_{3,4} = i$ the metric (3.11) (or its equivalent (3.15)) can be made Kähler. We are now in the realm of our earlier calculations of sec 2.1. We then expect that the background equations of motion will yield the following metric in type IIB:

$$\begin{aligned} ds^2 ~ &= \mathcal{D}(r)^2~dr^2 + \mathcal{D}(r)^{-2}~\Big(dz + Q_o~{\rm cot}~\theta_1~dx + Q_o~{\rm cot}~\theta_2~dy\Big)^2 ~ + \\ &+ \mathcal{D}(r)~(dy^2 + dx^2 + d\theta_2^2 + d\theta_1^2), \end{aligned} \tag{3.23}$$

where $\mathcal{D}(r) ~=~ 1 + Q_o~r$ is a linear function of r and we have written in terms of un-tilded coordinates to avoid clutter.

In the absence of an RR background, a Kähler (or a conformally Kähler) geometry generically breaks supersymmetry. Therefore only a non-Kähler deformation of the above background along with H_{NS} fluxes will preserve supersymmetry. We have already discussed the possibility of the existence of such a background from F-theory (in sec. 2.1), and here our ansatz for such a metric will be to deform the Kähler background (3.23) by switching on a non-trivial complex structure $\tau_{3,4}$ such that the final type IIB metric is

$$\begin{aligned} ds^2 ~ &= \mathcal{D}(r)^2~dr^2 + \mathcal{D}(r)^{-2}~\Big(dz + Q_o~{\rm cot}~\theta_1~dx + Q_o~{\rm cot}~\theta_2~dy\Big)^2 ~ + \\ &+ ~\mathcal{D}_1(r)~|dx + \tau_3~dy|^2 + \mathcal{D}_2(r)~|d\theta_1 + \tau_4~d\theta_2|^2, \end{aligned} \tag{3.24}$$

where the functional form for $\mathcal{D}_i(r)$ could in general be non-linear, and the metric of the two cycle on which we have wrapped NS5 will have the following form for $r \to 0$ (i.e in our local patch):

$$ds^2_{2-{\rm cycle}} ~ = ~ \Big(Q_o^2~{\rm cot}^2~\theta_2 + |\tau_3|^2\Big)~dy^2 ~ + ~ |\tau_4|^2~d\theta_2^2. \tag{3.25}$$

Therefore, once we know the local type IIB geometry, we can again use our sigma-model identification to go back to the heterotic side! Then the final local metric in the heterotic theory for the wrapped NS5 branes is given by the non-Kähler metric (3.24) with a torsion. After geometric transition we can infer the *full* global metric, which is derived in [28].

Another possibility is when we allow some of components of the B_{NS} fields to vanish. For example we could consider

$$b_{x\theta_2} \neq 0, \quad b_{y\theta_1} \neq 0, \quad b_{x\theta_1} \;=\; b_{y\theta_2} = 0. \tag{3.26}$$

In this case the previous simplifying relations between the d_i cannot be imposed, resulting in extra cross- terms in the metric. We can simplify the ensuing analysis a little by first observing that the B-fields are defined on compact spaces, and therefore are periodic. We can then write them in terms of angular coordinates θ and $\tilde{\theta}$ in the following way:

$$\theta \;=\; \arctan\left[\frac{b_{y\theta_1}\sqrt{d_1}}{\sqrt{d_6}}\right], \quad \tilde{\theta} \;=\; \arctan\left[\frac{b_{x\theta_2}\sqrt{d_2}}{|\tau_2|\sqrt{d_6}}\right], \tag{3.27}$$

where τ_2 represents the non-trivial complex structure discussed earlier in (3.1). In fact before geometric transition, the inherited complex structure is $\tau_2 = i$ (3.1). Here we would like to switch on a non-zero real part of τ_2 so as to simplify some of the following analysis. Of course to switch on such a complex structure we have to change the torsion a bit. Assuming that it is indeed possible to do this, we find that the following choice of τ_2 simplifies the metric quite a bit:

$$\frac{\text{Re }\tau_2}{|\tau_2|} \;=\; \frac{1}{2}\frac{d_4}{\sqrt{d_1 d_2}}\tan\theta\,\tan\tilde{\theta}. \tag{3.28}$$

The geometry of the two two-tori is very interesting now. In our previous case discussed above we see that the (θ_1, θ_2) and the (x, y) tori are decoupled (3.24). Now the (θ_1, θ_2) torus is non-trivially fibered over the other (x, y) torus[7]. One can work out the fibration precisely, and it turns out to have the following form:

$$ds^2_{\text{fib}} \;=\; d_6\sec^2\theta\Big[d\theta_1 + \sin 2\theta(a\;dx - b\;dy)\Big]^2 + d_6\;|\tau_2|^2\sec^2\tilde{\theta}\Big[d\theta_2 - \sin 2\tilde{\theta}(\tilde{a}\;dx - \tilde{b}\;dy)\Big]^2, \tag{3.29}$$

[7] Readers should observe that this is the simplest solution for this case. For our previous example, there could also be non-trivial fibration of the (θ_1, θ_2) torus over (x, y) base, but the metric presented in (3.24) is the simplest case.

which in fact would be more complicated if Re $\tau_2 = 0$. For Re τ_2 satisfying (3.28), there are no $d\theta_1 d\theta_2$ cross terms. The coefficients appearing in the metric are defined in terms of d_i as (we have taken d_4 here as one-half of the original choice):

$$\begin{aligned} a &= \frac{d_4}{4\sqrt{d_1 d_6}}; & b &= \frac{\sqrt{d_1}}{2\sqrt{d_6}}; \\ \tilde{b} &= \frac{d_4}{4\ |\tau_2|\sqrt{d_2 d_6}}; & \tilde{a} &= \frac{\sqrt{d_2}}{2\ |\tau_2|\sqrt{d_6}}. \end{aligned} \tag{3.30}$$

The base torus (x, y) is now no longer as simple as (3.13). In fact the original complex structures of (3.13) don't give the full metric. The (x, y) torus metric is

$$ds^2_{\rm xy} \ = \ |d\tilde{x} + \tau_3\ d\tilde{y}|^2 - \sigma_o\ |d\tilde{x} + \tau_5\ d\tilde{y}|^2, \tag{3.31}$$

where $a, \tilde{a}$ are defined in (3.30) and the tilde-coordinates $(\tilde{x}, \tilde{y})$ are scaled by d_2 from the original coordinates (x, y) given in (3.12). The shift σ_o in (3.31) is

$$\sigma_o \ = \ 4 d_6 d_2^{-1}\big[a^2\ \sin^2\ \theta + |\tau_2|^2\ \tilde{a}^2\ \sin^2\ \tilde{\theta}\big], \tag{3.32}$$

with $\tau_5 \equiv {\rm Re}\ \tau_5 + i\ {\rm Im}\ \tau_5$ defined in terms of the above variables as

$$\begin{aligned} {\rm Re}\ \tau_5 &= -\frac{ab\ \sin^2\ \theta + |\tau_2|^2\ \tilde{a}\tilde{b}\ \sin^2\ \tilde{\theta}}{a^2\ \sin^2\ \theta + |\tau_2|^2\ \tilde{a}^2\ \sin^2\ \tilde{\theta}}; \\ {\rm Im}\ \tau_5 &= \frac{(a\tilde{b} - \tilde{a}b){\rm sin}\ 2\theta\ \sin\ 2\tilde{\theta}}{d_6\big[a^2\ \sin^2\ \theta + |\tau_2|^2\ \tilde{a}^2\ \sin^2\ \tilde{\theta}\big]}, \end{aligned} \tag{3.33}$$

which together with (3.29) captures the full $\mathbf{T}^2 \otimes \mathbf{T}^2$ structure of the base where $\otimes$ denotes the non-trivial fibration of (θ_1, θ_2) torus on the (x, y) torus. The metric for our case now is not (3.11) but a more complicated one given as

$$ds^2 \ = \ ds^2_{\rm fib}\ +\ ds^2_{\rm xy}\ + d\tilde{z}^2 + dr^2, \tag{3.34}$$

where $ds^2_{\rm fib}$ and $ds^2_{\rm xy}$ are given above in (3.29) and (3.31) respectively. The solution (3.34), however, is still *not* the full metric with wrapped $NS5$ branes and torsion. This is of course similar to the case encountered earlier where (3.11) was not the full metric whereas (3.24) was. Here too our ansatz will be that the final heterotic metric can be written as

$$\begin{aligned} ds^2 \ &= \mathcal{H}(r)^2\ dr^2 + \mathcal{H}(r)^{-2}\ \Big(dz + Q_o\ \cot\ \theta_1\ dx + Q_o\ \cot\ \theta_2\ dy\Big)^2 + \\ &+\ \mathcal{H}_1(r)\ \Big(|dx\ +\ \tau_3\ dy|^2 - \sigma_o\ |dx + \tau_5\ dy|^2\Big)\ +\ \mathcal{H}_2(r)\ ds^2_{\rm fib}, \end{aligned} \tag{3.35}$$

where as before $\mathcal{H}(r)$ is a linear function in r, and $\mathcal{H}_i(r)$ could generically be non-linear. Similarly the metric of the two-cycle on which we have wrapped NS5 branes will be more complicated than the one derived before in (3.25), and is given by:

$$\begin{aligned} ds^2_{2-\text{cycle}} \; = & \Big[\mathcal{H}^{-2}\; Q_o^2 \cot^2\,\theta_2 + \mathcal{H}_1\; |\tau_3|^2 + (\mathcal{H}_2 - \mathcal{H}_1)(\tilde{d}_1\, \sin^2\,\theta + \tilde{d}_2\,\sin^2\,\tilde{\theta})\Big]\; dy^2\, + \\ & + \mathcal{H}_2\; d_6\; |\tau_2|^2\, \sec^2\,\tilde{\theta}\; d\theta_2^2, \end{aligned} \tag{3.36}$$

where $\tilde{d}_1 \equiv d_1^2, \tilde{d}_2 \equiv \frac{d_4^2}{4d_2}$. For our local patch where $r \to 0$ we expect (3.36) to reduce to

$$ds^2_{2-\text{cycle}} \; = \lim_{\epsilon\to 0}\, \Big[Q_o^2 \cot^2\theta_2 + |\tau_3|^2 + \epsilon(d_1' \sin^2\theta + d_2' \sin^2\tilde{\theta})\Big]\, dy^2 + d_6\, |\tau_2|^2 \sec^2\tilde{\theta}\, d\theta_2^2, \tag{3.37}$$

where we have assumed that $\epsilon = \mathcal{H}_2 - \mathcal{H}_1$ is a very small quantity. Thus (3.37) is almost similar to the metric of the two-cycle discussed earlier for (3.25), which means that the two-cycle doesn't change too much locally even if we consider different choices of the B fields. Once we know the two-cycle, a geometric transition will take us to the dual gravity theory whose *global* metric was derived earlier in [28].

The local heterotic geometry before geometric transition therefore has the following topology: the four-dimensional base $\mathcal{B}$ is a non-trivial T^2 fibration over another T^2. The first T^2 is parametrised by the coordinates (θ_1, θ_2) and the other T^2 is parametrised by (x, y) with a non-trivial complex structure. In addition to that there is an overall $U(1)$ fibration of dz over the base $\mathcal{B}$. Of course both the tori and the $U(1)$ directions are warped differently along the radial direction r and the local topology is of the form

$$\left(\mathbf{T}^2 \;\otimes\; \mathbf{T}^2\right) \;\otimes\; \mathbf{S}^1 \;\times\; \mathbb{R}^+ \tag{3.38}$$

where $\otimes$ denotes a non-trivial fibration, and the $NS5$ branes wrap two cycles in the manifold $\mathbf{T}^2 \;\otimes\; \mathbf{T}^2$. Geometrically the non-trivial fibration makes the metric non-Kähler, and the torsion is caused by the sources of the $NS5$ branes. We will discuss more details of this manifold and a family of them in sec. 3.3.

Alternatively one can see that there is a non-trivial $\mathbf{T}^3$ torus over the base (3.31). The metric of the $\mathbf{T}^3$ is

$$\begin{aligned} ds^2_{T^3} \; = & \;\mathcal{H}(r)^{-2}\big(dz + \alpha_1\cdot dx + \sigma_1 \cdot dy\big)^2 \; + \\ & + \mathcal{H}_3(r)\; \big(d\theta_1 + \alpha_2\cdot dx + \sigma_2\cdot dy\big)^2 + \mathcal{H}_4(r)\; \big(d\theta_2 + \alpha_3 \cdot dx + \sigma_3\cdot dy\big)^2, \end{aligned} \tag{3.39}$$

where $\mathcal{H}(r)$ is a linear function of r, and the other two warp factors are defined as

$$\mathcal{H}_3 \ = \ d_6 \ \mathcal{H}_2 \sec^2 \theta, \qquad \mathcal{H}_4 \ = \ d_6 \ |\tau_2|^2 \ \mathcal{H}_2 \sec^2 \tilde{\theta}, \tag{3.40}$$

while the warp factor $\mathcal{H}_2$ has already appeared in (3.35). The other coefficients α_i, σ_i can be easily extracted from (3.35) and (3.29). They are given by:

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} = \begin{pmatrix} \cot\theta_1 & a \sin 2\theta & -\tilde{a}\sin 2\tilde{\theta} \\ \cot\theta_2 & -b\sin 2\theta & \tilde{b}\sin 2\tilde{\theta} \end{pmatrix}, \tag{3.41}$$

where $(a, b, \tilde{a}, \tilde{b})$ are defined in (3.30). From the metric (3.39) we see that the $\mathbf{T}^3$ fibration is only approximate. The dz fibration also depends on the other (θ_1, θ_2) torus, and therefore a global geometry is more likely to be an extension of (3.38) instead of (3.39). One should also note that this $\mathbf{T}^3$ fibration is *not* the $\mathbf{T}^3$ fibration on which we can perform mirror transformation.

3.2. Analysis of the torsion classes

We would now like to classify the heterotic non–Kähler metric (3.35). Such non-Kähler backgrounds are conveniently classified in terms of their intrinsic torsion or their so–called torsion classes. These torsion classes correspond to the decomposition of the intrinsic torsion into $SU(3)$ representations, because four–dimensional supersymmetry requires the internal manifold to have an $SU(3)$ structure [73], [3]. See [1] for a rather mathematical discussion of manifolds with G–structure. These manifolds are characterized by a globally defined $SU(3)$ invariant spinor that is constant w.r.t. a torsional connection. This reduces the structure group of the six-dimensional manifold from $SO(6)$ to $SU(3)$ and the intrinsic torsion decomposes under $SU(3)$ into five classes, see e.g. [3], [2]: $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$.

The failure of the torsional connection to be the Levi–Civita connection is measured in the failure of fundamental 2–form and holomorphic 3–form to be closed. Defining a set of real vielbeins $\{e_i\}$ one can define an almost complex structure via a set of complex vielbeins $\{E_i\}$ as

$$E_1 = e_1 + i \ e_2 \ , \quad E_2 = e_3 + i \ e_4 \ , \quad E_3 = e_5 + i \ e_6, \tag{3.42}$$

which give rise to a (1,1)–form w.r.t. this almost complex structure

$$J = e_1 \wedge e_2 + e_3 \wedge e_4 + e_5 \wedge e_6 \, . \tag{3.43}$$

Similarly, one defines a holomorphic 3–form

$$\Omega = \Omega_+ + i\ \Omega_- = (e_1 + i\ e_2) \wedge (e_3 + i\ e_4) \wedge (e_5 + i\ e_6)\,, \tag{3.44}$$

where Ω_+ and Ω_- are the real and imaginary part of Ω, respectively. J and Ω fulfill the compatibility relations

$$\begin{aligned} J \wedge \Omega_+ &= J \wedge \Omega_- = 0; \\ \Omega_+ \wedge \Omega_- &= \frac{2}{3} J \wedge J \wedge J\,. \end{aligned} \tag{3.45}$$

The torsion classes $\mathcal{W}_i$ are then determined by the following equations

$$\begin{aligned} d\Omega_\pm \wedge J = \Omega_\pm \wedge dJ &= \mathcal{W}_1^\pm\ J \wedge J \wedge J, \\ d\Omega_\pm^{(2,2)} &= \mathcal{W}_1^\pm\ J \wedge J + \mathcal{W}_2^\pm \wedge J, \\ dJ^{(2,1)} &= (J \wedge \mathcal{W}_4)^{(2,1)} + \mathcal{W}_3\,, \end{aligned} \tag{3.46}$$

so $\mathcal{W}_1$ is given by two real numbers, $\mathcal{W}_1^+$ and $\mathcal{W}_1^-$. $\mathcal{W}_2$ is a (1,1) form and $\mathcal{W}_3$ is a (2,1) form. With the definition of the contraction

$$\rightharpoonup\ : \ \bigwedge{}^k\ T^* \otimes \bigwedge{}^n\ T^* \longrightarrow \bigwedge{}^{n-k}\ T^*, \tag{3.47}$$

and the convention $e_1 \wedge e_2 \ \rightharpoonup\ e_1 \wedge e_2 \wedge e_3 \wedge e_4 \ = \ e_3 \wedge e_4$ one defines

$$\begin{aligned} \mathcal{W}_4 &= \frac{1}{2}\ J \rightharpoonup dJ, \\ \mathcal{W}_5 &= \frac{1}{2}\ \Omega_+ \rightharpoonup d\Omega_+\,. \end{aligned} \tag{3.48}$$

We now want to study the intrinsic torsion of the heterotic metric (3.35). This metric can be brought into the form

$$\begin{aligned} ds^2 \ =& \ \mathcal{H}(r)^2\ dr^2 + \mathcal{H}(r)^{-2}\ \Big(dz + Q_o\ \cot\theta_1\ dx + Q_o\ \cot\theta_2\ dy\Big)^2 \\ &+\ \mathcal{H}_1(r)\ (1-\sigma_0)\ \ |dx\ +\ \tau_6\ dy|^2 \\ &+\ \mathcal{H}_2(r)\ d_6\ \Big(\sec^2\theta\Big[d\theta_1 + \sin 2\theta(a\ dx - b\ dy)\Big]^2 \\ &\quad + |\tau_2|^2 \sec^2\tilde\theta\Big[d\theta_2 - \sin 2\tilde\theta(\tilde a\ dx - \tilde b\ dy)\Big]^2\Big), \end{aligned} \tag{3.49}$$

where the new complex structure τ_6 of the (x,y)–torus is a function of τ_3, τ_5 and σ_0 and determined by

$$\mathrm{Re}\ \tau_6 \ = \ \frac{\mathrm{Re}\ \tau_3 - \sigma_0 \mathrm{Re}\ \tau_5}{1-\sigma_0}\ , \qquad |\tau_6|^2 \ = \ \frac{|\tau_3|^2 - \sigma_0|\tau_5|^2}{1-\sigma_0}\ . \tag{3.50}$$

The (θ_1, θ_2)–torus is non–trivially fibered over the (x, y)–torus, as mentioned before.

In the following we will assume that the original B–fields $b_{x\theta_2},\ b_{y\theta_1}$ (or equivalently $\theta,\ \tilde{\theta}$) are functions of r only. This translates into an r–dependence of $\tilde{a},\ \tilde{b},\ \sigma_0,\ \tau_2,\ \tau_5$ and τ_6, whereas the coefficients $d_i,\ a,\ b,\ Q_0$ and τ_3 remain constant. Since $\tilde{a},\ \tilde{b},\ \sigma_0$ and τ_5 are actually given through $|\tau_2|$ and $\theta,\ \tilde{\theta}$, we have only five independent functions left: $\mathcal{H}_1(r),\ \mathcal{H}_2(r),\ \theta,\ \tilde{\theta}$ and $|\tau_2|$ (but recall that τ_2 is related to θ and $\tilde{\theta}$ via (3.28)). $\mathcal{H}$ is linear in r and therefore completely determined by two real constants.

Due to its toroidal structure, there is a very natural choice for the complex structure on this metric. We define real vielbeins

$$\begin{aligned}
e_1 &= \mathcal{H}(r)\ dr\ , \qquad\qquad e_2 \ =\ \mathcal{H}(r)^{-1}\ \big(dz + Q_o\ \cot\ \theta_1\ dx + Q_o\ \cot\ \theta_2\ dy\big),\\
e_3 &= \sqrt{\mathcal{H}_1(r)\ (1-\sigma_0)}\ \ (dx\ +\ \mathrm{Re}\ \tau_6\ dy)\ , \qquad e_4 \ =\ \sqrt{\mathcal{H}_1(r)\ (1-\sigma_0)}\ \ \mathrm{Im}\ \tau_6\ dy,\\
e_5 &= \sqrt{\mathcal{H}_2(r)\ d_6}\ \ \sec\ \theta\ \Big[d\theta_1 + \sin\ 2\theta\ (a\ dx - b\ dy)\Big],\\
e_6 &= \sqrt{\mathcal{H}_2(r)\ d_6}\ \ |\tau_2|\ \sec\ \tilde{\theta}\ \Big[d\theta_2 - \sin\ 2\tilde{\theta}\ (\tilde{a}\ dx - \tilde{b}\ dy)\Big]
\end{aligned} \tag{3.51}$$

and the usual almost complex structure is induced by choosing complex vielbeins (3.42).

Let us first study $\mathcal{W}_4$ and $\mathcal{W}_5$. As it was shown in [3] supersymmetry for a heterotic solution with torsion requires $\mathcal{W}_1 = \mathcal{W}_2 = 0$ together with

$$\mathcal{W}_5\ =\ -2\ \mathcal{W}_4. \tag{3.52}$$

In terms of real vielbeins e_i one finds with the definition (3.48),

$$\begin{aligned}
\mathcal{W}_4 \ &=\ \frac{1}{2\mathcal{H}}\ \Big(\tan\theta\ \tilde{\theta}' + \tan\tilde{\theta}\ \tilde{\theta}' + \frac{\mathcal{H}_2'}{\mathcal{H}_2} + \frac{\mathcal{H}_1'}{\mathcal{H}_1} + \frac{(|\tau_2|)'}{|\tau_2|} + \frac{(\sigma_0)'}{\sigma_0 - 1} + \frac{(\mathrm{Im}\ \tau_6)'}{\mathrm{Im}\ \tau_6}\\
&\qquad -\ \frac{b\ Q_0\ \csc^2\theta_1\ \ \sin 2\theta + \tilde{a}\ Q_0\ \csc^2\theta_2\ \ \sin 2\tilde{\theta}}{\mathcal{H}_1(\sigma_0 - 1)\ \mathrm{Im}\ \tau_6}\Big)\ \ e_1;\\
\mathcal{W}_5 \ &=\ -\ \frac{1}{2\mathcal{H}}\ \Big(\tan\theta\ \theta' + \tan\tilde{\theta}\ \tilde{\theta}' + \frac{\mathcal{H}_2'}{\mathcal{H}_2} + \frac{\mathcal{H}_1'}{\mathcal{H}_1} + \frac{(|\tau_2|)'}{|\tau_2|} + \frac{(\sigma_0)'}{\sigma_0 - 1} + \frac{(\mathrm{Im}\ \tau_6)'}{\mathrm{Im}\ \tau_6} - \frac{\mathcal{H}'}{\mathcal{H}}\Big)\ \ e_1\\
&\qquad +\ \frac{(\mathrm{Re}\ \tau_6)'}{2\mathcal{H}\ \mathrm{Im}\ \tau_6}\ \ e_2.
\end{aligned} \tag{3.53}$$

Here, the prime indicates a derivative w.r.t. r. We will not attempt to solve the supersymmetry conditions fully. We would find five differential equations for the five independent functions mentioned above when imposing (3.52) and $\mathcal{W}_1 = \mathcal{W}_2 = 0$. But they are highly non-trivial, even if we assume linear r–dependence for $\mathcal{H}$ and $\mathcal{H}_{1,2}$. However, it is immediately obvious that for (3.52) to be fulfilled the part of $\mathcal{W}_5$ that is proportional to e_2 has to vanish. We therefore find the first non–trivial constraint

$$\text{Re}\ \tau_6 \ = \ \text{const.} \tag{3.54}$$

This means that the (x, y) torus is not quite a square torus and would have been square if Re τ_6 were identically zero. On the other hand (3.54) translates into

$$\begin{aligned} \text{const} \ &= \ -\frac{1}{2}\Bigg[4d_1 d_2 - d_4^2 + \frac{\sin^2 2\theta\ \sin^2 2\tilde{\theta}(-d_2 + 4d_6\ P(r))(\tilde{a}b - a\tilde{b})^2}{d_2 d_6\ P(r)} \\ &\quad - \ 16 d_2\ P(r)\left(d_1 - \frac{d_4^2}{4d_2} - \frac{d_4\ \tilde{P}(r)}{8\sqrt{d_2}\ P(r)}\right)\Bigg]^{1/2} \left(d_2^2 - 4d_2 d_6\ P(r)\right)^{-1}, \end{aligned} \tag{3.55}$$

where we have introduced the abbreviations

$$P(r) \ = \ a^2 \sin^2\theta + (\tilde{a})^2 |\tau_2|^2 \sin^2\tilde{\theta}\ , \qquad \tilde{P}(r) \ = \ ab\ \sin^2\theta + \tilde{a}\tilde{b}\ |\tau_2|^2\ \sin^2\tilde{\theta}. \tag{3.56}$$

To fulfill this condition, the r–dependences of θ, $\tilde{\theta}$, $|\tau_2|$, $\tilde{a}$ and $\tilde{b}$ need to be carefully balanced.

The remaining torsion classes are evaluated using only the constraint (3.54). For the real and imaginary part of $\mathcal{W}_1$ one obtains

$$\begin{aligned} \mathcal{W}_1^+ \ &= \Bigg(Q_0\ \csc^2\theta_1\ \cos\theta\ \text{Re}\tau_6 + 2d_6\mathcal{H}_2\ \Big[a\cos 2\theta\ \sec\theta\ \text{Im}\tau_6\ \tilde{\theta}' \\ &\qquad + |\tau_2|\Big(\tilde{b}\ \sin\tilde{\theta} - \tilde{a}'\ \text{Re}\tau_6\ \sin 2\tilde{\theta} + \cos 2\tilde{\theta}\ \sec\tilde{\theta}\ \left(\tilde{b} - \tilde{a}\ \text{Re}\ \tau_6\right)\ \tilde{\theta}'\Big)\Big]\Bigg)\times \\ &\quad \times\ \left(6\mathcal{H}\ \text{Im}\ \tau_6\ \sqrt{d_6\ \mathcal{H}_1\mathcal{H}_2\ (1-\sigma_0)}\right)^{-1}; \\ \mathcal{W}_1^- \ &= \Big(|\tau_2|\ \cos\theta\ \left[Q_0\ \csc^2\theta_1\ \text{Im}\tau_6 + 2d_6\mathcal{H}_2(b - a\ \text{Re}\tau_6)\cos 2\theta\ \sec^2\theta\ \theta'\right] \\ &\qquad - d_6\mathcal{H}|\tau_2|^2\text{Im}\tau_6\ \sec\tilde{\theta}\ \left[\tilde{a}'\ \sin 2\tilde{\theta} + 2\tilde{a}\ \cos 2\tilde{\theta}\ \tilde{\theta}'\right] - Q_0\ \csc^2\theta_2\ \cos\tilde{\theta}\Big)\times \\ &\quad \times\ \left(6\mathcal{H}|\tau_2|\ \text{Im}\ \tau_6\ \sqrt{d_6\ \mathcal{H}_1\mathcal{H}_2\ (1-\sigma_0)}\right)^{-1}. \end{aligned} \tag{3.57}$$

It turns out that $\mathcal{W}_2$ has only two nonzero components; $E_2 \wedge \overline{E}_3$ and $E_3 \wedge \overline{E}_2$, which are furthermore identical for the $\mathcal{W}_2^+$–part and have opposite sign for the $\mathcal{W}_2^-$ part. Their precise values are determined to be

$$
\begin{aligned}
\mathcal{W}_2^+ &= \Big[\mathrm{Im}\tau_6\,\Big(-|\tau_2|' + |\tau_2|\big(\tan\theta\;\theta' - \tan\tilde\theta\;\tilde\theta'\big)\Big) + |\tau_2|\;(\mathrm{Im}\tau_6)'\Big]\;\frac{i\,(E_2\wedge\overline{E}_3 + E_3\wedge\overline{E}_2)}{4\mathcal{H}|\tau_2|\;\mathrm{Im}\tau_6},\\
\mathcal{W}_2^- &= \Big[\mathrm{Im}\tau_6\,\Big(-|\tau_2|' + |\tau_2|\big(\tan\theta\;\theta' - \tan\tilde\theta\;\tilde\theta'\big)\Big) - |\tau_2|\;(\mathrm{Im}\tau_6)'\Big]\;\frac{(E_2\wedge\overline{E}_3 - E_3\wedge\overline{E}_2)}{4\mathcal{H}|\tau_2|\;\mathrm{Im}\tau_6}.
\end{aligned}
\tag{3.58}
$$

Note that these two-forms do not just differ by an overall factor: there is a sign difference in the $(\mathrm{Im}\tau_6)'$ term as well. The last torsion class $\mathcal{W}_3$ is given by the (2,1)–form

$$
\begin{aligned}
\mathcal{W}_3 \;&= \frac{X_1(\overline{\tau}_6) + i\;X_2(\overline{\tau}_6)}{X_3}\;E_1\wedge E_2\wedge\overline{E}_3 \;+\; \frac{X_1(\tau_6) - i\;X_2(\tau_6)}{X_3}\;E_1\wedge E_3\wedge\overline{E}_2\\
&+\frac{X_1(\overline{\tau}_6) - i\;X_2(\overline{\tau}_6)}{X_3}\;E_2\wedge E_3\wedge\overline{E}_1,
\end{aligned}
\tag{3.59}
$$

where $\overline{\tau}_6 = \mathrm{Re}\tau_6 - i\;\mathrm{Im}\tau_6$ and we have introduced

$$
\begin{aligned}
X_1(\tau_6) &= Q_0\;\csc^2\theta_2\;\;\cos\tilde\theta + 2d_6\mathcal{H}_2|\tau_2|\;\;\cos 2\theta\;\sec\theta\;\theta'\;(a\;\tau_6 - b),\\
X_2(\tau_6) &= Q_0\;\csc^2\theta_1\;|\tau_2|\cos\theta\;\tau_6 - d_6\mathcal{H}_2|\tau_2|^2\sec\tilde\theta\;\big[\sin 2\tilde\theta\;(\tilde a'\tau_6 - \tilde b') + 2\cos 2\tilde\theta\;(\tilde a\tilde\theta'\;\tau_6 - \tilde b)\big],\\
X_3 &= 8\mathcal{H}|\tau_2|\;\sqrt{d_6\mathcal{H}_1\mathcal{H}_2(1-\sigma_0)}\;\mathrm{Im}\;\tau_6.
\end{aligned}
\tag{3.60}
$$

These non–vanishing torsion classes show that our heterotic background (3.35) will in general be non–Kähler. The supersymmetry conditions $\mathcal{W}_1 = \mathcal{W}_2 = 0$ and $\mathcal{W}_5 \;=\; -2\;\mathcal{W}_4$ result in five differential equations that are given through

$$
\begin{aligned}
\mathcal{W}_1^+ \;&= \mathcal{W}_1^- \;=\; 0,\\
\mathcal{W}_2^+|_{E_2\wedge\overline{E}_3} \;&= \mathcal{W}_2^-|_{E_2\wedge\overline{E}_3} \;=\; 0,\\
\mathcal{W}_5|_{e_1} \;&=\; -\,2\mathcal{W}_4|_{e_1}.
\end{aligned}
\tag{3.61}
$$

Solving these together with the constraint (3.55) should in principle allow for a determination of the functions $\mathcal{H}_1(r),\ \mathcal{H}_2(r),\ \theta,\ \tilde\theta$ and $|\tau_2|$, but we were not able to find an analytic solution. In the following section, we will provide a detailed mathematical analysis of these non-Kähler manifolds.

3.3. A family of non-Kähler manifolds

In this section we will first quickly describe the global geometry corresponding to (3.35) (or (3.49) in an expanded form), then we give the analysis that leads to it, and finally we analyze the geometry.

The non-Kähler complex threefold $\mathbf{X}$ is desribed in terms of a non-Kähler complex surface $\mathbf{S}$ called a *primary Kodaira surface* which was already known to mathematicians. The surface $\mathbf{S}$ has a non-trivial holomorphic fibration over $\mathbf{T}^2$ with $\mathbf{T}^2$ fiber characterized as being twisted by translations in a topologically non-trivial manner. This will be made more precise below. The desired complex threefold $\mathbf{X}$ is a nontrivial holomorphic $\mathbf{C}^*$ fibration over $\mathbf{S}$. It admits a nowhere vanishing holomorphic 3 form Ω. This geometry will be described in more detail following the analysis that led to it.

We know that $\mathbf{X}$ is built from a holomorphic base $\mathbf{T}^2$ in two steps: first construct a complex surface $\mathbf{S}$ by two $\mathbf{S}^1$ fibrations over $\mathbf{T}^2$, leading to a holomorphic $\mathbf{T}^2$ fibration over $\mathbf{T}^2$. Then we take an $\mathbf{S}^1$ fibration Y over $\mathbf{S}$ and take $X = Y \times \mathbf{R}$. The holomorphic $\mathbf{T}^2$ fibrations over $\mathbf{T}^2$ are completely classified. A convenient reference for the results are in [74] Section V.5.

Let $\mathbf{B}$ denote the base $\mathbf{T}^2$ with its holomorphic structure as an elliptic curve $\mathbf{C}/\Lambda$, where Λ is a lattice $\mathbf{Z} \oplus \tau_B \mathbf{Z}$. Let $\mathbf{E}$ denote the fiber $\mathbf{T}^2$ with its holomorphic structure as an elliptic curve $\mathbf{C}/L$, where L is a lattice $\mathbf{Z} \oplus \tau_E \mathbf{Z}$.

We need to study holomorphic automorphisms of $\mathbf{E}$ in order to build non-trivial holomorphic bundles with fiber $\mathbf{E}$. The translation automorphisms of $\mathbf{E}$ can be identified with $\mathbf{E}$ itself by associating to $e \in \mathbf{E}$ the translation automorphism t_e of $\mathbf{E}$ defined by $t_e(z) = z + e$. Letting $\mathrm{Aut}(\mathbf{E})$ denote the group of holomorphic automorphisms of $\mathbf{E}$, and $\mathbf{E} \subset \mathrm{Aut}(\mathbf{E})$ the translation subgroup just described, then it is well-known that $(\mathrm{Aut}(\mathbf{E}))/\mathbf{E}$ is a finite group $\mathbf{Z}_n$. Usually $n = 2$ and these automorphisms are just the $\mathbf{Z}_2$ subgroup generated by the inversion $z \mapsto -z$ of $\mathbf{E}$, but n can be larger if

$$\mathbf{E} \simeq \mathbf{C}/(\mathbf{Z} \oplus i\mathbf{Z}) \qquad \text{or} \qquad \mathbf{E} \simeq \mathbf{C}/(\mathbf{Z} \oplus \omega\mathbf{Z}) \tag{3.62}$$

with $\omega = \exp(\pi i/3)$. For the first elliptic curve in (3.62), we have $n = 4$ with the $\mathbf{Z}_4$ generated by the automorphism $z \mapsto iz$. For the second elliptic curve in (3.62), we have $n = 6$ with the $\mathbf{Z}_6$ generated by the automorphism $z \mapsto \omega z$. This description makes $\mathrm{Aut}(\mathbf{E})$ into a disconnected 1 dimensional complex manifold, identified with n disjoint copies of $\mathbf{E}$.

We now turn to the description of non-trivial holomorphic $\mathbf{E}$ fibrations over $\mathbf{B}$. The surface $\mathbf{S}$ is constructed by choosing open sets $U_i \subset \mathbf{B}$ covering the base $\mathbf{B}$ and gluing the trivial products $U_i \times \mathbf{E}$ and $U_j \times \mathbf{E}$ using nontrivial holomorphic automorphisms of $(U_i \cap U_j) \times \mathbf{E}$.

We introduce w as the coordinate on $\mathbf{B}$. When studying $U_i \times \mathbf{E}$, we will use the coordinates (w, z_i). In other words, we continue to use the coordinate w on $U_i \subset \mathbf{B}$, and modify notation slightly by using z_i instead of z as the coordinate on $\mathbf{E}$. The introduction of this subscript allows us to describe a gluing $U_i \times \mathbf{E}$ with $U_j \times \mathbf{E}$ by identifying the common $(U_i \cap U_j) \times \mathbf{E}$ subset via an identification which necessarily takes the form

$$(w, z_j) = (w, (\rho_{ij}(w))(z_i)) \tag{3.63}$$

Here, $\rho_{ij}(w)$ is a holomorphic automorphism of $\mathbf{E}$ depending holomorphically on w. In other words, the mapping $\rho_{ij} : U_i \cap U_j \to \mathrm{Aut}(\mathbf{E})$ is holomorphic.

Let $\mathcal{A}(\mathbf{E})$ denote the trivial holomorphic fiber bundle over $\mathbf{B}$ with fiber $\mathrm{Aut}(\mathbf{E})$, i.e. $\mathcal{A}(\mathbf{E}) = \mathbf{B} \times \mathrm{Aut}(\mathbf{E})$ as a complex manifold. Then ρ_{ij} is a holomorphic section of the bundle $\mathcal{A}(\mathbf{E})$ over the open set $U_i \cap U_j$.

We have the obvious compatibility condition:

$$\rho_{jk}(w) \circ \rho_{ij}(w) = \rho_{ik}(w), \tag{3.64}$$

valid for $w \in U_i \cap U_j \cap U_k$. In other words, $\{\rho_{ij}\}$ is a Cech cocycle representing a cohomology class

$$\rho \in H^1(\mathbf{B}, \mathcal{A}(\mathbf{E})). \tag{3.65}$$

The quotient map $\pi : \mathrm{Aut}(E) \to \mathbf{Z}_n$ induces a class $\pi(\rho) \in H^1(\mathbf{B}, \mathbf{Z}_n)$.[8]

The first relevant result is that if $\pi(\rho)$ is non-trivial, then $\mathbf{S}$ is Kähler (actually projective algebraic). Such an algebraic surface is called a *hyperelliptic surface*. Details are in [74] (pp147–8). Since the $\mathbf{S}$ that we need is non-Kähler, we require that $\pi(\rho) \in H^1(\mathbf{B}, \mathbf{Z}_n)$ is the trivial cohomology class.

Let $\mathcal{E}$ be the trivial holomorphic bundle over $\mathbf{B}$ with fiber $\mathbf{E}$. The exact sequence

$$0 \to \mathcal{E} \to \mathcal{A}(\mathbf{E}) \to \mathbf{Z}_n \to 0 \tag{3.66}$$

determines a corresponding exact sequence of cohomology groups

$$H^1(\mathbf{B}, \mathcal{E}) \to H^1(\mathbf{B}, \mathcal{A}(\mathbf{E})) \to H^1(\mathbf{B}, \mathbf{Z}_n). \tag{3.67}$$

Since the second map takes ρ to $\pi(\rho)$, assumed to be trivial, from this sequence we see that ρ arises from a cohomology class in $H^1(\mathbf{B}, \mathcal{E})$. In other words, we can and will assume

[8] $\pi(\rho)$ is a natural shorthand for the Cech cohomology class represented by the cocycle whose value over $U_i \cap U_j$ is $\pi(\rho_{ij}(w)) \in \mathbf{Z}_n$.

that the ρ_{ij} take values in the translation subgroup $\mathbf{E} \subset \mathrm{Aut}(\mathbf{E})$. We let $\sigma_{ij}(w) \in \mathbf{E}$ by the element of $\mathbf{E}$ corresponding to these translations, i.e.

$$\rho_{ij}(w) = t_{\sigma_{ij}(w)}, \tag{3.68}$$

where t_e continues to denote the translation automorphism by an element $e \in \mathbf{E}$. Then the compatibility condition (3.64) implies the compatibility condition

$$\sigma_{jk}(w) + \sigma_{ij}(w) = \sigma_{ik}(w), \tag{3.69}$$

valid for $w \in U_i \cap U_j \cap U_k$. The condition (3.69) says that the σ_{ij} define a cohomology class $\sigma \in H^1(\mathbf{B}, \mathcal{E})$.

From the description of $\mathbf{E}$ as the quotient of $\mathbf{C}$ by the lattice $L = \mathbf{Z} + \tau_E \mathbf{Z}$, we have an exact sequence

$$0 \to L \to \mathbf{C} \to \mathbf{E} \to 0 \tag{3.70}$$

which determines the cohomology coboundary mapping

$$H^1(\mathbf{B}, \mathcal{E}) \to H^2(\mathbf{B}, L). \tag{3.71}$$

The resulting class in $H^2(\mathbf{B}, L)$ will be denoted by $c(\rho)$. It is a generalization of the well-known Chern class of a $U(1)$ bundle; in fact, if one chooses an isomorphism $L \simeq \mathbf{Z}^2$ (e.g. the one determined by 1 and τ_E), then $H^2(\mathbf{B}, L) \simeq H^2(\mathbf{B}, \mathbf{Z})^2$ and $c(\rho)$ is identified with the Chern classes of the two S^1 bundles used to construct S.

In [74] (page 146), it is shown that if $c(\rho) = 0$, then S is a complex torus, hence Kähler. Thus we require that $c(\rho) \neq 0$, or equivalently that the T^2 bundle is topologically nontrivial. This is the situation of a *primary Kodaira surface* considered in [74] (pp 146–7). The invariants are

$$H^1(\mathbf{S}, \mathbf{Z}) = \mathbf{Z}^3, \quad H^2(\mathbf{S}, \mathbf{Z}) = \mathbf{Z}^4 \text{ or } \mathbf{Z}^4 \oplus \mathbf{Z}_m. \tag{3.72}$$

These invariants can also be computed by the Gysin sequence as has been used in [16]. Note that since $H^1(\mathbf{S}, \mathbf{R})$ is odd-dimensional, $\mathbf{S}$ is not Kähler.

Furthermore, $\mathbf{S}$ admits a nowhere vanishing holomorphic 2-form.[9] This can be made explicit. From (3.63) and (3.68) we have the coordinate transformation

$$z_j = z_i + \sigma_{ij}(w). \tag{3.73}$$

[9] In [74] this is described as the triviality of the canonical bundle.

It follows immediately from (3.73) that

$$dz_j \wedge dw = dz_i \wedge dw. \tag{3.74}$$

Thus the holomorphic 2-forms $dz_i \wedge dw$ in $U_i \times \mathbf{E}$ agree on their overlaps and patch together to give a well-defined holomorphic 2-form $dz \wedge dw$ on $\mathbf{S}$, even though dz is not a well-defined 1-form.

Next, we consider a non-trivial S^1 bundle Y over $\mathbf{S}$; then finally put $X = Y \times \mathbf{R}$. Thus the fiber S^1 is promoted to $S^1 \times \mathbf{R} \simeq \mathbf{C}^*$.[10] We study holomorphic $\mathbf{C}^*$ bundles $\mathbf{X}$ over $\mathbf{S}$. We explicitly assume that the $\mathbf{C}^*$ bundle structure is a principal bundle.[11]

We cover $\mathbf{S}$ by open sets V_α and build $\mathbf{X}$ by gluing together the open sets $V_\alpha \times \mathbf{C}^*$. Since $\mathbf{S}$ is non-Kähler, methods for constructing metrics on $\mathbf{X}$ from the metric on $\mathbf{S}$ will not produce a Kähler metric. Presumably $\mathbf{X}$ does not admit any exotic Kähler metrics, but we have not definitively ruled out that possibility.

Let t_α be the $\mathbf{C}^*$ coordinate in $V_\alpha \times \mathbf{C}^*$. The principal bundle ansatz is that $\mathbf{C}^*$ multiplications are used to perform the gluing, i.e. that the coordinates are related by

$$t_\beta = \kappa_{\alpha\beta}(w, z) t_\alpha, \tag{3.75}$$

where $\kappa_{\alpha\beta} : V_\alpha \cap V_\beta \to \mathbf{C}^*$ is holomorphic. Then (3.75) implies that

$$dw \wedge dz \wedge \frac{dt_\beta}{t_\beta} = dw \wedge dz \wedge \frac{dt_\alpha}{t_\alpha} \tag{3.76}$$

in $(V_\alpha \cap V_\beta) \times \mathbf{C}^*$, so these holomorphic 3-forms patch to give a nowhere vanishing holomorphic 3-form $\Omega = dw \wedge dz \wedge dt/t$.

It remains to check the existence of nontrivial holomorphic principal $\mathbf{C}^*$ fibrations over $\mathbf{S}$. The transition functions $\kappa_{\alpha\beta}$ satisfy

$$\kappa_{\beta\gamma}\kappa_{\alpha\beta} = \kappa_{\alpha\gamma} \tag{3.77}$$

in $V_\alpha \cap V_\beta \cap V_\gamma$, so define a cohomology class $\gamma \in H^1(\mathbf{S}, \mathcal{O}^*_{\mathbf{S}})$. Here $\mathcal{O}^*_{\mathbf{S}}$ denotes the sheaf of nowhere vanishing holomorphic functions on arbitrary open subsets of $\mathbf{S}$, and has been

[10] Note that $\mathbf{C}^*$ has a unique holomorphic structure up to isomorphism, so we use the standard one.

[11] Our duality chain indicates a similar structure.

introduced since $\kappa_{\alpha\beta}$ is a holomorphic section of $\mathcal{O}^*_{\mathbf{S}}$ over $V_\alpha \cap V_\beta$. So we only have to show that $H^1(\mathbf{S}, \mathcal{O}^*_{\mathbf{S}})$ is nontrivial.

For this purpose, we have the exponential sequence of sheaves on $\mathbf{S}$

$$0 \to \mathbf{Z} \to \mathcal{O}_{\mathbf{S}} \to \mathcal{O}^*_{\mathbf{S}} \to 0, \tag{3.78}$$

where $\mathbf{Z}$ denotes the sheaf of locally constant integer-valued functions and $\mathcal{O}_{\mathbf{S}}$ are the holomorphic functions. The first non-trivial map in (3.78) is the inclusion, and the second map takes a holomorphic function f to $\exp(2\pi i f)$.

The cohomology sequence associated to (3.78) includes the segment

$$\cdots \to H^1(\mathbf{S}, \mathbf{Z}) \to H^1(\mathbf{S}, \mathcal{O}_{\mathbf{S}}) \to H^1(\mathbf{S}, \mathcal{O}^*_{\mathbf{S}}) \to H^2(\mathbf{S}, \mathbf{Z}) \to H^2(\mathbf{S}, \mathcal{O}_{\mathbf{S}}) \to \cdots, \tag{3.79}$$

so that $H^1(\mathbf{S}, \mathcal{O}^*_{\mathbf{S}})$ contains $H^1(\mathbf{S}, \mathcal{O}_{\mathbf{S}})/H^1(\mathbf{S}, \mathbf{Z})$ as a subgroup.

In [74], $H^1(\mathbf{S}, \mathcal{O}_{\mathbf{S}}) \simeq H^{0,1}(\mathbf{S})$[12] has been computed to be $\mathbf{C}^2$. In our situation, we can even explicitly exhibit two independent $\bar{\partial}$-closed $(0,1)$ forms on $\mathbf{S}$: the global form $d\bar{w}$ coming from the base $\mathbf{B}$, and the form $d\bar{z}$ on the fiber, which is globally well-defined by (3.73) and the holomorphicity of σ_{ij}.

Combining this calculation with (3.72), we see that $H^1(\mathbf{S}, \mathcal{O}^*_{\mathbf{S}})$ contains a subgroup of the form $\mathbf{C}^2/\mathbf{Z}^3$, which is certainly nontrivial as claimed.

We can also show that we have holomorphic $\mathbf{C}^*$ fibrations with topologically nontrivial structure. The topology can be computed from the coboundary mapping $\delta : H^1(\mathbf{S}, \mathcal{O}^*_{\mathbf{S}}) \to H^2(\mathbf{S}, \mathbf{Z})$ from (3.79). The image in $H^2(\mathbf{S}, \mathbf{Z})$ of the cohomology class in $H^1(\mathbf{S}, \mathcal{O}^*_{\mathbf{S}})$ representing our holomorphic fibration is just the chern class of the original S^1 bundle. So if we specify a non-trivial class $c \in H^2(\mathbf{S}, \mathbf{Z})$ corresponding to a topologically nontrivial S^1 bundle and want to know if the corresponding $S^1 \times \mathbf{R} = \mathbf{C}^*$ fibration admits a holomorphic principal bundle structure, we ask if c is in the image of δ. By (3.79), this is equivalent to asking if the image of c in $H^2(\mathbf{S}, \mathcal{O}_S)$ vanishes. But $H^2(\mathbf{S}, \mathcal{O}_S)$ is a vector space, so if, for example, we are in the case with torsion, $H^2(\mathbf{S}, \mathbf{Z}) = \mathbf{Z}^4 \oplus \mathbf{Z}_m$ as in (3.72), the torsion classes $c \in \mathbf{Z}_m$ must map to 0 as complex vector spaces have no torsion classes. Hence these classes are in the image of δ and the corresponding $S^1 \times \mathbf{R}$ fibration supports a holomorphic $\mathbf{C}^*$ structure. We conclude that there are certainly examples of complex threefolds $\mathbf{X}$ with the properties dictated by the duality chain.

[12] The Dolbeault isomorphism used here is valid for non-Kähler manifolds.

In summary, the fibration structured dictated by the duality chain led us to construct specific manifolds $\mathbf{X}$ with (integrable) complex structures. Not only do we show that these exist, but we see that these manifolds have nowhere vanishing holomorphic 3-forms, something that we didn't require in the construction, a good check. We leave the study of the metric and gauge bundle to future work.

We close this section with a brief description of the topology of $\mathbf{X}$. Since $\mathbf{R}$ is contractible, it follows that $\mathbf{X}$ is homotopic to Y, hence

$$H^*(\mathbf{X}, \mathbf{Z}) \simeq H^*(Y, \mathbf{Z}). \tag{3.80}$$

The Gysin sequence for the S^1 bundle $\mathbf{Y}$ reads

$$\cdots \to H^i(\mathbf{S}, \mathbf{Z}) \to H^{i+2}(\mathbf{S}, \mathbf{Z}) \to H^{i+2}(Y, \mathbf{Z}) \to H^{i+1}(\mathbf{S}, \mathbf{Z}) \to \cdots \tag{3.81}$$

The first map in (3.81) is cup product with the first chern class c_1 of the S^1 bundle $\mathbf{Y}$. The second map in (3.81) is the pullback map associated with the projection $\mathbf{Y} \to \mathbf{S}$.

Without knowing more than the nontriviality of c_1, there is not enough information in (3.81) to even determine the cohomology ranks. So that we can say something more definite, let us suppose that the S^1 fibration is general enough that the first map in c_1 has the maximum possible rank, namely the minimum of the ranks of $H^i(\mathbf{S}, \mathbf{Z})$ and $H^{i+2}(\mathbf{S}, \mathbf{Z})$. From (3.72), (3.80), (3.81), and Poincaré duality on $\mathbf{S}$, it is computed in this situation that, ignoring possible torsion,

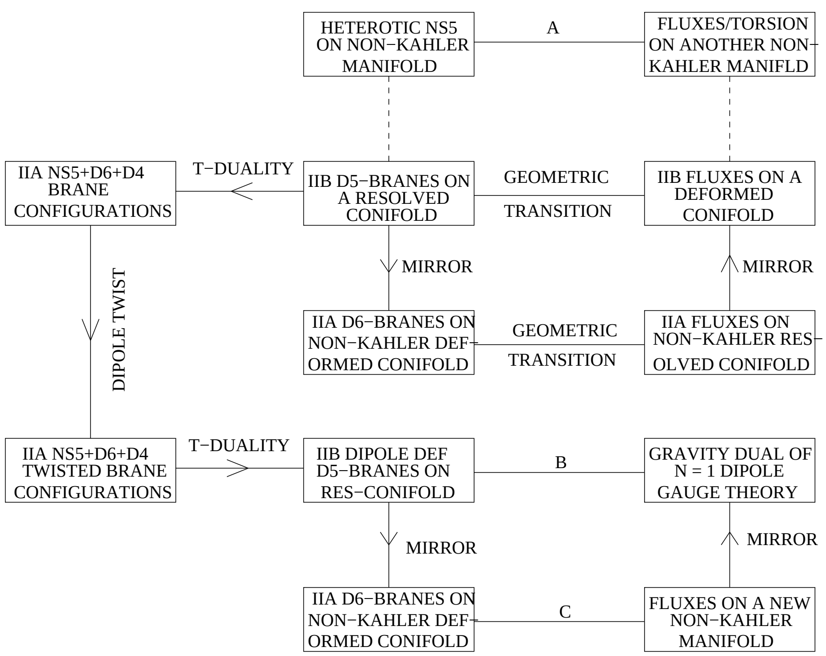
$$H^i(\mathbf{X}, \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{i=0, 5} \\ \mathbf{Z}^3 & \text{i=1,2,3,4} \\ 0 & \text{otherwise} \end{cases} \tag{3.82}$$

4. Dipole deformations in the geometric transition setup

There is one important issue that we have overlooked for some time in the setup of geometric transition in type IIB theory. This is the appearance of dipole deformations in these theories. Recall that due to inherent orientifold action, the allowed choices of the B fields are given by (2.53). The B_{NS} field is thus oriented along $\mathcal{B}_{y\theta_1}$ and therefore has one leg along the $D5$ branes wrapped on the two-torus (y, θ_2). As we now know from [34] this situation is ripe for dipole theories (see also [75] and the second reference of [76]). The $\mathcal{B}_{y\theta_1}$ gives rise to dipole deformations of our geometric transition setup from the five-brane point of view. These dipoles are therefore *not* visible in the IR of our gauge theory, and their presence is fully manifested once we go to the far UV of the theory where we expect six-dimensional gauge theory. For our case, assuming that we choose a scale where we do not integrate out the dipole degrees of freedom, we can then ask two questions:

• Can we calculate the precise supergravity metric for our case now? Recall that now our ingredients will be seven-branes, wrapped five-branes and the B fields generating the dipole deformations *on* a non-trivial background that locally looks like a Kähler resolved conifold.

• Can we see what happens to this background after we perform a geometric transition? Clearly there would be a gravitational dual to this theory because (a) it is $\mathcal{N} = 1$ gauge theory, and (b) it has dipole deformations. So far we have been pursuing both cases individually in [26], [27], and [28] for the gravity duals of confining gauge theories; and in [34], [76] for the gravity duals of dipole theories. We would like to know what happens now when we combine these[13].



In **figure 4** above we have represented the whole set-up. The middle set of boxes (and the operations therein) have already been dealt with in [26], [28]. The dotted line that takes us to the heterotic theory is not *a priori* connected to type IIB because we used a local orientifolding operation to go to the heterotic side and then moved away from the

[13] Recall that in [28] we computed the mirror type IIA picture, where the dipole deformations in type IIB theory results in specific non-Kähler deformations in type IIA.

orientifold point to study the global story. The gauge theory (i.e the NS5 branes side) is basically what we addressed above. The gravity dual was studied in [28]. Whether the two sides are connected by a possible geometric transition is still unknown, and so we have used **A** to represent this.

The part of the figure that takes us from the wrapped $D5$ branes picture to an equivalent IIA brane configuration is the starting point of the dipole deformation. In fact this procedure addresses the first of the two questions mentioned above, namely, determining the supergravity solution for our wrapped D5 brane configuration with dipole deformations on the world-volume of the branes. This is already a complicated enterprise, because our dipole theory is now embedded in a much more non-trivial background. Due to the complexity of the problem, we will address this in two steps:

• First we will see how our type IIB metric ansatz (2.27) with warp factors (2.28) changes when we incorporate dipole deformations on the background without $D5$ (or $D7$)-branes. We will only be able to work this order-by-order in the B field $\mathcal{B}_{y\theta_1} \equiv b$ by treating the dipole deformation as a perturbation.

• Secondly, we will insert back all the branes in our framework and study the final dipole deformed metric. The analysis will be done up to some orders in r and b as above, so that this would capture the essential feature of the whole story. The usual constraint of an order-by-order expansion is that we cannot tell how higher order terms modify the result. Nevertheless, we will be able to pursue both avenues in enough detail so that the final picture can be presented clearly and unambigously.

The second question of determining the gravity duals of these new theories is represented in the figure above by the lines joining the last two boxes with letters **B** and **C** as the underlying operation (as yet unknown). Clearly, after we go to the IIA brane configuration, and then do a dipole twist followed by a T-duality we *do not* come back to the same configuration. This is of course expected from our earlier studies (see [34]) as we go to dipole theories. This is a very interesting scenario because it opens up, for the first time, the possibility of studying gravity duals of dipole theories in the setting of the geometric transition! In fact the whole set of transformations of [26] could possibly be done to get the gravity duals of these theories. We can ask many questions here:

• How do we see the non-locality of dipole theories in the gravity duals of these theories?

• In the type IIA mirror configuration, we map directly to a non-Kähler manifold instead of another dipole theory. What happens to the dipoles of the original theory?

• What are the operations **B** and **C**? Are they in any way connected to the geometric transitions?
• What happens in M-theory? How do we go to the gravity duals in IIA using M-theory? Is there an equivalent operation to the flop here too?

In this paper we will only be able to address the supergravity solution for the wrapped D5 branes with dipole deformations (this is not the gravity dual!) following the two-step procedure that we mentioned above. The rest of the questions will be addressed in the sequel to this paper [41].

Before we go ahead with the dipole story, let us mention one more thing regarding the dipoles studied earlier in [34]. The dipole theories that we studied before were associated with vanishing beta functions, i.e. conformal theories[14]. The gravity duals of these theories were therefore determined from the *near-horizon* geometries [34] exactly in the same way as for other CFTs [77]. On the other hand, the theories that we are studying here are confining gauge theories and have non-zero beta functions. The gravity duals of these theories follow a somewhat different route as we saw earlier in [23], [24], [26], [28], [27]. Once we make a dipole deformation to these theories the gravity duals follow yet other different routes mentioned as operations **B** and **C** in the figure above. These details will be relegated to the sequel to this paper.

4.1. Supergravity solution without branes

As we mentioned earlier, our basic point is to treat the dipole deformations perturbatively. In previous sections we have managed to study the local metric from analysing the equations of motion, without carefully considering the backreactions of fluxes and seven branes on the geometry. Here we would like to study the local metric when we put in everything like the branes and fluxes along with a non-trivial background geometry.

From the very look of it, this is a pretty complicated problem. Therefore we will attack it in a few simple steps: First we analyse the background with back-reactions from B-fields. These B-fields would eventually be responsible for the dipole deformations on the wrapped D5 branes. Next we insert back the D7 branes in the geometry by including their back-reactions. And finally we create bound states of D5 branes on a single D7 brane by switching on first Chern classes. This way we will have explicit supersymmetric solutions for the system. Of course this is not the only way to get susy solutions here. As we

[14] Even though they have a length scale, the dipole length!

discussed in much detail earlier, F-theory allows us to study a supersymmetric solution with separated D5s, D7s and primitive fluxes. Once we separate the D7 branes we are not bound to remain at the orientifold point, and there we can have all possible components of the B-fields. The local supergravity solution of the system with B-fields along the five-brane directions has been given earlier in (2.49) without carefully considering the full back-reactions. Here we want to see possible corrections to this metric when we consider all the branes and fluxes.

We begin by first removing the seven-branes, but keeping the fluxes. We shall assume that the metric ansatz for our case looks similar to (2.27) i.e

$$ds^2_{\mathcal{M}} \;=\; \mathcal{A}\; dr^2 + \mathcal{B}\;(dz + f_1\; dx + f_2\; dy)^2\; + (\mathcal{C}\; d\theta_1^2 + \mathcal{D}\; dx^2) + (\mathcal{E}\; d\theta_2^2 + \mathcal{F}\; dy^2), \tag{4.1}$$

with the coefficients having the same expansion as (2.28) although we might have to go beyond r^2 terms in (2.28) to see the full back-reaction. We will deal with these details as we go on. As a starter we need new combinations of the warp factors as

$$\mathcal{E}(1 - \mathcal{F}) \;=\; D_0 - D_1\; r - D_2\; r^2 - D_3\; r^2 + ... \tag{4.2}$$

where $\mathcal{E}, \mathcal{F}$ have the same expansion as (2.28) before. The coefficients D_i in the expansion above are defined as

$$\begin{aligned} D_0 \;&=\; 0, \quad D_1 \;=\; \frac{\alpha_6}{\mathcal{F}_6}\; r, \quad D_2 \;=\; \frac{1}{\mathcal{F}_6}\left(\beta_6 + \frac{\alpha_5\alpha_6}{\mathcal{F}_5(r_0)}\right), \\ D_3 \;&=\; \frac{1}{\mathcal{F}_6(r_0)}\left(\gamma_6 \;+\; \frac{\alpha_5\beta_6}{\mathcal{F}_5(r_0)} \;+\; \frac{\beta_5\alpha_6}{\mathcal{F}_5(r_0)}\right), \end{aligned} \tag{4.3}$$

where γ_6 is an $\mathcal{O}(r^3)$ term in $\mathcal{F}$ (2.28). In addition to that, the coefficients $\alpha_{5,6}$ and $\beta_{5,6}$ have the same equivalence relations as in (2.34), although – and this is very crucial – $\alpha_{3,4}$ and $\beta_{3,4}$ are no longer related by (2.34) because of the back-reactions of the B-fields. The coefficients $\gamma_{5,6}$ are expected to satisfy the following approximate relation

$$\frac{\gamma_5}{\mathcal{F}_5(r_0)} \;-\; \frac{\gamma_6}{\mathcal{F}_6(r_0)} \;\approx\; 0 \tag{4.4}$$

where now γ_5 is an $\mathcal{O}(r^3)$ term in $\mathcal{E}$ (2.28), much like γ_6 above. In [41] we will discuss a toy example where an exact equality in (4.4) (and also in (2.34)) is realised.

The reader may notice that we haven't said anything too new so far. Let us now switch on a B_{NS} field $\mathcal{B}_{y\theta_1} \equiv b$ and define an expansion of the form

$$\kappa_o \ \equiv \ 1 \ + \ a_1 \ b^2 \ \mathcal{E}(1-\mathcal{E}) \ + \ a_2 \ b^4 \ \mathcal{E}^2(1-\mathcal{E})^2 \ + \ ..... \tag{4.5}$$

with constant coefficients $a_{1,2}$ that we will be kept arbitrary in this paper. The expansion κ_o will help us see the corrections to the equations (2.35) due to the B-field b. For this we need, instead of (2.41), a more involved structure,

$$\kappa_o \mathcal{D}\mathcal{E} \ = \ 1 \ + \ E_0 r \ + \ E_1 r^2 \ + \ E_2 r^3 \ + \ E_3 r^4 \ + \ \mathcal{O}(r^5), \tag{4.6}$$

where we are assuming that the E_3 coefficient is well defined. The various E_i can be written in terms of $(\alpha_4, \alpha_5, \alpha_6)$, $(\beta_4, \beta_5, \beta_6)$, $(\gamma_4, \gamma_5, \gamma_6)$ and b as

$$\begin{aligned}
E_0 \ &= \ \frac{\alpha_5}{\mathcal{F}_5(r_0)} + \frac{\alpha_4}{\mathcal{F}_4(r_0)} - \frac{a_1 b^2 \alpha_6}{\mathcal{F}_6(r_0)}, \\
E_1 \ &= \ \frac{a_2 b^4 \alpha_6 - a_1 b^2 \beta_6}{\mathcal{F}_6(r_0)} + \frac{\beta_5}{\mathcal{F}_5(r_0)} + \frac{\beta_4}{\mathcal{F}_4(r_0)} - \frac{2 a_1 b^2 \alpha_5 \alpha_6}{\mathcal{F}_5(r_0)\mathcal{F}_6(r_0)} \ + \\
& - \frac{a_1 b^2 \alpha_6}{\mathcal{F}_6(r_0)} + \frac{\alpha_5 \alpha_4}{\mathcal{F}_5(r_0)\mathcal{F}_4(r_0)}, \\
E_2 \ &= \ \frac{a_2 b^4 \alpha_5 \alpha_6}{\mathcal{F}_5(r_0)\mathcal{F}_6(r_0)} + \frac{a_2 b^4 \alpha_4 \alpha_6}{\mathcal{F}_4(r_0)\mathcal{F}_6(r_0)} - \frac{2 a_1 b^2 \alpha_5 \beta_6}{\mathcal{F}_5(r_0)\mathcal{F}_6(r_0)} - \frac{a_1 b^2 \beta_6 \alpha_4}{\mathcal{F}_6(r_0)\mathcal{F}_4(r_0)} \ - \\
& - \frac{a_1 b^2 \alpha_5^2 \alpha_6}{\mathcal{F}_5^2(r_0)\mathcal{F}_6(r_0)} - \frac{2 a_1 b^2 \alpha_5 \alpha_6 \alpha_4}{\mathcal{F}_5(r_0)\mathcal{F}_6(r_0)\mathcal{F}_4(r_0)} - \frac{a_1 b^2 \gamma_6}{\mathcal{F}_6(r_0)} - \frac{a_1 b^2 \beta_5 \alpha_6}{\mathcal{F}_5(r_0)\mathcal{F}_6(r_0)} \ + \\
& - \frac{a_1 b^2 \beta_4 \alpha_6}{\mathcal{F}_4(r_0)\mathcal{F}_6(r_0)} + \frac{\gamma_5}{\mathcal{F}_5(r_0)} + \frac{\gamma_4}{\mathcal{F}_4(r_0)} + \frac{\alpha_4 \beta_5 + \alpha_4 \gamma_5 + \alpha_5 \beta_4}{\mathcal{F}_5(r_0)\mathcal{F}_4(r_0)}, \\
E_3 \ &= \ \frac{a_2 b^4 (\alpha_5 \alpha_6 + \alpha_6 \beta_5) - a_1 b^2 (2\alpha_5 \gamma_6 + 2\beta_5 \beta_6 + \gamma_5 \gamma_6 + \alpha_6 \gamma_5)}{\mathcal{F}_5(r_0)\mathcal{F}_6(r_0)} \ + \\
& + \frac{a_2 b^4 \beta_6 - a_1 b^2 \gamma_6}{\mathcal{F}_6(r_0)} - \frac{a_1 b^2 (\gamma_6 \alpha_4 + \alpha_6 \gamma_4)}{\mathcal{F}_6(r_0)\mathcal{F}_4(r_0)} - \frac{a_1 b^2 (\alpha_5^2 \beta_6 + 2\beta_5 \alpha_6 \alpha_5)}{\mathcal{F}_5^2(r_0)\mathcal{F}_6(r_0)} \ + \\
& - \frac{a_1 b^2 (2\alpha_5 \beta_6 \alpha_4 + 2\beta_5 \alpha_6 \alpha_4 + \alpha_6 \alpha_4 \gamma_5 + 2\alpha_5 \alpha_6 \beta_4) - a_2 b^4 \alpha_6 \alpha_5 \alpha_4}{\mathcal{F}_4(r_0)\mathcal{F}_5(r_0)\mathcal{F}_6(r_0)} \ + \\
& - \frac{a_1 b^2 \alpha_5^2 \alpha_6 \alpha_4}{\mathcal{F}_4(r_0)\mathcal{F}_5^2(r_0)\mathcal{F}_6(r_0)} + \frac{a_2 b^4 \alpha_6 \beta_4 - a_1 b^2 \beta_6 \beta_4}{\mathcal{F}_4(r_0)\mathcal{F}_6(r_0)} + \frac{\beta_4 \beta_5 + \gamma_4 \alpha_5}{\mathcal{F}_4(r_0)\mathcal{F}_6(r_0)},
\end{aligned} \tag{4.7}$$

where we see that even in the absence of a B field we expect higher order corrections to

(2.31) given here as

$$
\begin{aligned}
E_0^0 &= C_0, \quad E_1^0 = C_1, \quad E_3^0 = C_3 + \frac{\gamma_3\alpha_5}{\mathcal{F}_3(r_0)\mathcal{F}_6(r_0)}, \\
E_2^0 &= C_2 + \frac{\gamma_5}{\mathcal{F}_5(r_0)} + \frac{\gamma_3}{\mathcal{F}_3(r_0)} + \frac{\alpha_3\gamma_5}{\mathcal{F}_5(r_0)\mathcal{F}_3(r_0)},
\end{aligned}
\tag{4.8}
$$

with $C_3 = \frac{\beta_3\beta_5}{\mathcal{F}_3\mathcal{F}_5}$, and $E_i^0 \equiv \lim_{b\to 0} E_i$; and using (2.34). We see that the higher order corrections typically arise from $\gamma_i, i = 3, 5, 6$, terms in the absence of B fields.

Now to make connections with the B-fields and the coefficients of the proposed metric (4.1) we need to first define the coefficients to higher orders in r than what we took earlier in (2.28). Let us first consider the coefficient $\mathcal{B}$ in (4.1). This is given by

$$
\mathcal{B} = 1 + \frac{\alpha_2}{\mathcal{F}_2(r_0)}r + \frac{\beta_2}{\mathcal{F}_2(r_0)}r^2 + \frac{\gamma_2}{\mathcal{F}_2(r_0)}r^3 + \frac{\delta_2}{\mathcal{F}_2(r_0)}r^4 + \mathcal{O}(r^5). \tag{4.9}
$$

As one would expect, the connection between these coefficients and the E_i defined above is much more involved here than (2.35). This is of course expected, as the B-fields will back-react on the geometry and distort it from the simple form (2.44) (where we didn't consider the back-reactions carefully). As we will see, the distortion is order by order in the parameter b, and so we will not be too far from our original choice of metric (2.44). However our initial expectation of α_2, β_2 in (2.35) changes to

$$
\begin{aligned}
\frac{\alpha_2}{\mathcal{F}_2(r_0)} + \frac{\alpha_5}{\mathcal{F}_5(r_0)} + \frac{\alpha_4}{\mathcal{F}_4(r_0)} &= \frac{a_1 b^2\alpha_6}{\mathcal{F}_6(r_0)}, \\
\frac{\beta_2}{\mathcal{F}_2(r_0)} + \frac{\beta_5}{\mathcal{F}_5(r_0)} + \frac{\beta_4}{\mathcal{F}_4(r_0)} &= \frac{\alpha_5^2}{\mathcal{F}_5^2(r_0)} + \frac{\alpha_4^2}{\mathcal{F}_4^2(r_0)} + \frac{\alpha_5\alpha_3}{\mathcal{F}_5(r_0)\mathcal{F}_4(r_0)} + \\
&\quad - \frac{a_2 b^4\alpha_6 - a_1 b^2(\beta_6 + \alpha_6)}{\mathcal{F}_6(r_0)} + \frac{a_1^2 b^4\alpha_6^2}{\mathcal{F}_6^2(r_0)} - \frac{2a_1 b^2\alpha_4\alpha_6}{\mathcal{F}_4(r_0)\mathcal{F}_6(r_0)},
\end{aligned}
\tag{4.10}
$$

which in the limit $b \to 0$ starts to look like (2.35) once we incorporate the identifications in (2.34).

The other two coefficients γ_2 and δ_2 are too complicated to be written in terms of other coefficients of the metric (4.1). Therefore we will use the E_i's in (4.7) to write them down. Unfortunately the analysis turns out to be very involved, and only under some simplifying assumptions have we been able to find the following relations between the coefficients:

$$
\begin{aligned}
\frac{\gamma_2}{\mathcal{F}_2(r_0)} + E_2 + E_0^3 &= 2E_0E_1, \\
\frac{\delta_2}{\mathcal{F}_2(r_0)} + E_3 + 3E_0^2E_1 &= E_0^4 + 2E_0E_2 + E_1^2,
\end{aligned}
\tag{4.11}
$$

where one could write the values of E_i to get more direct relations. It will be a formidable exercise to disentangle any information out of this. Fortunately, this is not the complete story. We can find more relations between the coefficients in (4.1) to decipher the structure of the metric in terms of at least one or more warp factors (and the b field). One such set of connections is an immediate modification of (2.33) as

$$\begin{aligned} \frac{\alpha_1}{\mathcal{F}_1(r_0)} - \frac{\alpha_4}{\mathcal{F}_4(r_0)} - \frac{\alpha_5}{\mathcal{F}_5(r_0)} &= 0; \\ \frac{\beta_1}{\mathcal{F}_1(r_0)} - \frac{\beta_5}{\mathcal{F}_5(r_0)} - \frac{\beta_4}{\mathcal{F}_4(r_0)} &= \frac{\alpha_4\alpha_5}{\mathcal{F}_4(r_0)\mathcal{F}_5(r_0)}, \end{aligned} \tag{4.12}$$

where, when we apply (2.34) we get back (2.33) from above. Of course we cannot apply (2.34) to this case when we have a non-trivial b factor, and so (4.12) gives rise to new connections between the various coefficients.

So far we could get relations for all the expansions in (2.28) except $(\alpha_3, \beta_3, ...)$. Since the first line of (2.34) is not enough to get the relations, we have to see how (2.34) is corrected by b. Again the analysis is done order by order in b, and we see the following relations emerging from our calculations to correct our original evaluation of (2.34):

$$\begin{aligned} \frac{\alpha_3}{\mathcal{F}_3(r_0)} - \frac{\alpha_4}{\mathcal{F}_4(r_0)} &= F_0, \\ \frac{\beta_3}{\mathcal{F}_3(r_0)} - \frac{\beta_4}{\mathcal{F}_4(r_0)} &= \frac{F_0\alpha_4}{\mathcal{F}_4(r_0)} + F_0^2 - F_1, \end{aligned} \tag{4.13}$$

with F_0, F_1(as expected) dependent on the b field in the following way:

$$F_0 = \frac{a_1 b^2 \alpha_6}{\mathcal{F}_6(r_0)}, \qquad F_1 = \frac{a_2 b^4 \alpha_6}{\mathcal{F}_6(r_0)} - a_1 b^2 \left(\frac{\beta_6}{\mathcal{F}_6(r_0)} + \frac{\alpha_5\alpha_6}{\mathcal{F}_5(r_0)\mathcal{F}_6(r_0)}\right), \tag{4.14}$$

where both F_0, F_1 vanish when $b \to 0$ resulting in getting (2.34) from (4.13). One can also see that up to possible constants $a_{1,2}$ the corrections to (2.34) are known in powers of b. Although these corrections are evaluated using some approximations, they will be helpful later to fix the precise back-reactions due to b on the geometry. We also see that the corrections are dependent on α_6, β_5, etc. This may look a little counter-intuitive but will become clearer later when we fix the geometry. And finally, the other coefficients in

(2.28) are related as

$$\begin{aligned}
\frac{\gamma_3}{\mathcal{F}_3(r_0)} - \frac{\gamma_4}{\mathcal{F}_4(r_0)} &= \frac{F_0\beta_4}{\mathcal{F}_4(r_0)} + \frac{F_0^2\alpha_4}{\mathcal{F}_4(r_0)} - \frac{F_1\alpha_4}{\mathcal{F}_4(r_0)} - 2F_0F_1 + F_0^3 + F_2,\\
\frac{\delta_3}{\mathcal{F}_3(r_0)} - \frac{\delta_4}{\mathcal{F}_4(r_0)} &= \frac{F_0\gamma_4}{\mathcal{F}_4(r_0)} + \frac{F_0^2\beta_4}{\mathcal{F}_4(r_0)} - \frac{F_1\beta_4}{\mathcal{F}_4(r_0)} - \frac{(2F_0F_1 - F_0^3 - F_2)\alpha_4}{\mathcal{F}_4(r_0)} + \\
&\quad + 2F_0F_2 - F_3 + F_1^2 - 3F_0^2F_1 + F_0^4,
\end{aligned} \tag{4.15}$$

where we could go beyond these orders; but that will not be necessary for our purposes. We also see that the expansions are defined in terms of F_0, F_1 and two new terms F_2 and F_3. They are defined as

$$\begin{aligned}
F_2 &= a_1 b^2\left(\frac{\gamma_6}{\mathcal{F}_6(r_0)} + \frac{\alpha_5\beta_6 + \beta_5\alpha_6}{\mathcal{F}_5(r_0)\mathcal{F}_6(r_0)}\right),\\
F_3 &= \frac{a_2 b^4\beta_6}{\mathcal{F}_6(r_0)} + \frac{a_2 b^4\alpha_5\beta_6 - a_1 b^2(\alpha_5\gamma_6 + \beta_5\beta_6 + \gamma_5\alpha_6)}{\mathcal{F}_5(r_0)\mathcal{F}_6(r_0)} - \frac{a_1 b^2\gamma_6}{\mathcal{F}_6(r_0)}.
\end{aligned} \tag{4.16}$$

The above set of relations should be enough to get some relations between the warp factors in the metric (4.1). The first thing to look for are the complex structures of the two tori (y, θ_2) and (x, θ_1). The complex structures are different from the ones that we calculated earlier in (2.36). In fact we don't even expect them to be identical. They are:

$$dz_1 = dx + i\tau_{(1)}d\theta_1, \qquad dz_2 = dy + i\tau_{(2)}d\theta_2, \tag{4.17}$$

where $\tau_{(i)}$ are real numbers. It turns out that only $\tau_{(1)}$ is affected by the b field, and not $\tau_{(2)}$ (we will provide a reason later). Therefore $\tau_{(1)}$ is more involved than the other, and is given here by

$$\begin{aligned}
\tau_{(1)} &= 1 + \frac{F_0 r}{2} + \left(\frac{3F_0^2}{8} - \frac{F_1}{2}\right) r^2 + \left(\frac{F_2}{2} + \frac{5F_0^3}{16} - \frac{3F_0F_1}{4}\right) r^3 + \\
&\quad + \left(\frac{3F_1^2}{8} - \frac{F_3}{2} + \frac{3F_0F_2}{4} - \frac{15F_0^2F_1}{16} + \frac{35F_0^4}{128}\right) r^4,
\end{aligned} \tag{4.18}$$

where F_i are defined above as (4.14) and (4.16). One can easily see that in the absence of b field $\tau_{(1)}$ is just the identity (up to the orders that we consider), and in general this is given approximately by

$$\tau_{(1)} \approx \frac{1}{\sqrt{\kappa_o}} + \mathcal{O}(r^5), \tag{4.19}$$

where κ_o is given by the expansion (4.5). The above relation is only approximate as we have worked only up to $\mathcal{O}(r^4)$. For higher powers of r, or even large r, we need more detailed analysis. It is easy to see even for non-zero b that

$$\lim_{r\to 0} \tau_{(1)} \;=\; 1 \;+\; \mathcal{O}(r^5), \tag{4.20}$$

which gives a square (x, θ_1) torus at the far IR[15]. For the other complex structure of the (y, θ_2) torus we find

$$\begin{aligned}\tau^2_{(2)} \;&= 1 + \mathcal{F}_6(r_0)^{-1}\Big[\Big(\alpha_6 - G_0\mathcal{F}_6(r_0)\Big)\, r + \Big(G_1\mathcal{F}_6(r_0) - \alpha_6 G_0 + \beta_6\Big)\, r^2 \,+ \\ &+ \Big(\alpha_6 G_1 - G_2\mathcal{F}_6(r_0) - \gamma_6 G_0 + \gamma_6\Big)\, r^3 + \Big(\beta_6 G_1 - \alpha_6 G_2 - \gamma_6 G_0 + \delta_6\Big)\, r^4\Big],\end{aligned} \tag{4.21}$$

where we have defined G_i as

$$\begin{aligned} G_0 &= \frac{\alpha_5}{\mathcal{F}_5(r_0)}, \qquad G_1 = \frac{\alpha_5^2}{\mathcal{F}_5^2(r_0)} - \frac{\beta_5}{\mathcal{F}_5(r_0)}, \\ G_2 &= \frac{\gamma_5}{\mathcal{F}_5(r_0)} - \frac{2\alpha_5\beta_5}{\mathcal{F}_5^2(r_0)} + \frac{\alpha_5^3}{\mathcal{F}_5^3(r_0)}, \\ G_3 &= -\frac{\delta_5}{\mathcal{F}_5(r_0)} + \frac{\beta_5^2}{\mathcal{F}_5^2(r_0)} + \frac{2\alpha_5\beta_5}{\mathcal{F}_5^2(r_0)} - \frac{3\alpha_5^2\beta_5}{\mathcal{F}_5^3(r_0)} + \frac{\alpha_5^4}{\mathcal{F}_5^4(r_0)}, \end{aligned} \tag{4.22}$$

and all the variables in the above relations have already been defined. We also see some interesting consequences when we apply the second line of (2.34) to (4.21). The complex structure $\tau_{(2)}$ becomes

$$\tau_{(2)} \;=\; 1 + \mathcal{O}(r^5) \tag{4.23}$$

even for small but finite r. This means that the (y, θ_2) torus is exactly square at far IR, but the (x, θ_1) torus is only approximately square at far IR. At finite r, $\tau_{(2)}$ remains square, whereas $\tau_{(1)}$ receives b dependent corrections (see footnote above).

The conclusions about the complex structures that we gave above are not in any way unexpected. Combining the relation (4.13) with the second line of (2.34), we can reproduce

[15] A more appropriate result would be to consider $\tau_{(1)} = 1 + r\tilde{b}^2$ instead of just $\tau_{(1)} = 1$ where $\tilde{b} = \sqrt{\frac{a_1 b^2 \alpha_6}{\mathcal{F}_6(r_0)}}$ is the effective B-field. For small r this tells us how the b field affects the complex structure of the (x, θ_1) torus.

both the complex structures as evaluated above. What is interesting however, is that now we can write the two tori metric completely in terms of the $\tau_{(i)}$ in (4.23) and (4.19) as:

$$ds^2_{\rm tori} \;=\; {\rm x}_1\;|dy + {\rm i}d\theta_2|^2 + {\rm x}_2\;|dx + {\rm i}\kappa_o^{-\frac{1}{2}}d\theta_1|^2, \tag{4.24}$$

where κ_o is defined in (4.5) and ${\rm x}_{1,2}$ are unknown functions that have to be determined from the expansion above. In the far IR, $ds^2_{\rm tori}$ becomes the metric for two square tori with some r dependent coefficients. The b field simply distorts one of the tori so that it scales in some particular way as we move along the radial direction.

It is now easy to determine ${\rm x}_{1,2}$ from the expansion above. All we require is to represent them as some series like:

$${\rm x}_1 = 1 + \sum_{i=1} {\rm x}_{(1i)}r^i, \qquad {\rm x}_2 = 1 + \sum_{i=1} {\rm x}_{(2i)}r^i, \tag{4.25}$$

with the generic terms ${\rm x}_{(ai)} \neq 0$ for $a = 1, 2; i = 1, 2, ...$ The set of steps required to get to the final answer is to first evaluate the quantity $\frac{{\rm x}_2}{\kappa_o}$ and secondly compare these results to the relations (4.13) and (4.15). For ${\rm x}_1$ we can compare the metrics (4.1), (4.24) with (2.34). We will not show these analyses here[16], but readers can easily verify the following relations:

$${\rm x}_1 - \mathcal{E} \;=\; \mathcal{O}(r^5), \qquad {\rm x}_2 - \mathcal{D} \;=\; \mathcal{O}(r^5), \tag{4.26}$$

which specifies the metric (4.24) at least up to $\mathcal{O}(r^5)$ in the expansion above.

Once we have a relation like (4.26), the rest follows rather straightforwardly. The structure that we are alluding to is almost like (2.38) but is a little more complicated. For the present case we have:

$$\begin{aligned} &\mathcal{A} - \mathcal{D}\cdot\mathcal{E} \;=\; \mathcal{O}(r^5), \\ &\mathcal{B} - \frac{\kappa_o^{-1}\cdot\mathcal{D}^{-1}\cdot\mathcal{F}^{-1}}{\big(1 + b_0\;\cot\;\langle\theta_1\rangle\big)\big(1 + c_0\;\cot\;\langle\theta_2\rangle\big)} \;=\; \mathcal{O}(r^5), \end{aligned} \tag{4.27}$$

which differ from (2.38) in a crucial way. The above relations could be simplified further to take into account the complex structures of the two tori. Assuming b_0, c_0 to be very small, an obvious simplification occurs when in (4.27) we have the second relation modified to

$$\mathcal{B} - \kappa_o^{-1}\cdot\mathcal{D}^{-1}\cdot\mathcal{E}^{-1} \;=\; \mathcal{O}(r^5), \tag{4.28}$$

[16] One would require the expansion $\kappa_o^{-1} = 1 + F_0 r + (F_0^2 - F_1)r^2 - (F_2 + F_0^3 - 2F_0F_1)r^3 + (F_1^2 - F_3 + F_0^4 + 2F_0F_2 - 3F_0^2F_1)r^4$ with F_i defined in (4.14) and (4.16) to do the analysis.

where κ_o^{-1} expansion was given earlier as a footnote. We also see that the combination of (4.27) and (4.28) is close to the structure that we had earlier. This is good, because it means that our earlier choice of metric still survives possible dipole deformations. Of course at the far IR we shouldn't detect any observable effects of the dipoles, so this is not too surprising. In fact at the IR there could be further simplification coming from the fact that we are at small r. One such simplification is to look for the behavior of:

$$|\mathrm{x}_1 - \mathrm{x}_2| \;=\; \sum_i \; |\mathrm{x}_{(1i)} - \mathrm{x}_{(2i)}| \; r^i \tag{4.29}$$

which clearly is very small at $r \to 0$. What happens for finite r? In the absence of a b field, every term on the RHS of (4.29) for $i \leq 5$ vanishes. We will assume that this continues to hold even in the presence of the b field because dipole deformations will change results only in the far UV and not in IR. This would naturally then imply

$$|\mathrm{x}_1 - \mathrm{x}_2| \;=\; \mathcal{O}(r^5) \tag{4.30}$$

up to the order that we made our analysis so far. This simplifies (4.24). But this is not all. A few more simplifications follow immediately (we give only a partial analysis):

$$\begin{aligned}
\frac{\alpha_1}{\mathcal{F}_1(r_0)} &= \frac{2\alpha_6}{\mathcal{F}_6(r_0)}, \quad \frac{\beta_1}{\mathcal{F}_1(r_0)} = \frac{\alpha_6^2}{\mathcal{F}_6^2(r_0)} + \frac{2\beta_6}{\mathcal{F}_6(r_0)}, \quad \frac{\alpha_2}{\mathcal{F}_2(r_0)} = \frac{(a_1 b^2 - 2)\alpha_6}{\mathcal{F}_6(r_0)}, \\
\frac{\beta_2}{\mathcal{F}_2(r_0)} &= \frac{(3 - a_1^2 b^4 - 2a_1 b^2)\alpha_6^2}{\mathcal{F}_6^2(r_0)} - \frac{(a_2 b^4 + a_1 b^2)\alpha_6}{\mathcal{F}_6(r_0)} - \frac{(2 + a_1 b^2)\beta_6}{\mathcal{F}_6(r_0)}, \\
\frac{\alpha_3}{\mathcal{F}_3(r_0)} &= \frac{(1 + a_1 b^2)\alpha_6}{\mathcal{F}_6(r_0)}, \quad \frac{\beta_3}{\mathcal{F}_2(r_0)} = \frac{(1 + a_1 b^2)\beta_6}{\mathcal{F}_6(r_0)} + \frac{a_1^2 b^4 \alpha_6^2}{\mathcal{F}_6^2(r_0)} - \frac{a_2 b^4 \alpha_6}{\mathcal{F}_6(r_0)},
\end{aligned} \tag{4.31}$$

and more involved relations for (γ_i, δ_i) $i = 1, 2, 3$ in terms of $(\alpha_6, \beta_6, \gamma_6, \delta_6)$. Taking the complex coordinates for the two base tori to be

$$dz_1 \;=\; dx + \frac{\mathrm{i}}{\sqrt{\kappa_o}}\, d\theta_1, \qquad dz_2 \;=\; dy + \mathrm{i}\; d\theta_2, \tag{4.32}$$

the final metric is a simple modification of (2.44) that we had earlier:

$$\begin{aligned}
ds^2_{\mathcal{M}} &= \mathcal{F}(r)^2\; dr^2 + \kappa_o^{-1} \mathcal{F}(r)^{-2} \; \Big(dz + f_1(r, \theta_1)\; dx + f_2(r, \theta_2)\; dy\Big)^2 + \\
&\quad + \mathcal{F}(r)\; |dz_1|^2 + \mathcal{F}(r)\; |dz_2|^2,
\end{aligned} \tag{4.33}$$

which is as good as it gets because this is very close to (2.44); at far IR we expect this to coincide with (2.44). Whether this could be possible needs to be worked out now. We therefore require a way to evaluate

- The warp factor $\mathcal{F}(r)$, and
- The functions $f_1(r, \theta_1)$ and $f_2(r, \theta_2)$

to complete this side of the story. The crucial difference between the present metric and (2.44), other than the appearance of κ_o, is that both $f_{1,2}$ could be functions of r also. The base tori, as we observed earlier, are almost square and are deformed a little bit by the b field.

4.2. Dipole deformations and decoupling of the KK states

The second part of the story is to introduce the seven-branes to our background (4.33). These will back-react on the metric (4.33) to change the geometry. We will show that this is not difficult to work out. In addition to that, we will find that because of the seven-branes there will now be a non-trivial axion-dilaton switched on.

The third and the final part of the story is to bring back the D5 branes. We have already discussed a consistent way to do this: construct the D5 branes as bound states on a single D7 brane! This will guarantee a fully supersymmetric background with non-trivial fluxes on a non-trivial metric. Of course this is not the *only* way to have a supersymmetric background with seven-branes, $D5$ branes and fluxes. From F-theory, discussed earlier and in [26], [27], [28] we know that using F-theory we can have a fully consistent background with separated D5s and D7s (along with primitive fluxes). So the background that we construct in this section is clearly not the most generic; although it is simple enough to illustrate all the important ingredients of our analysis.

Before we move ahead to determine the warp factor, f_i and the branes, we want to make some comments on the field theory interpretation. One of the main difficulties in dealing with supergravity duals to confining field theories is to decouple the KK masses from the scale of the SUSY field theory. This is a crucial thing because we have wrapped D5 branes on a non-trivial $\mathbf{P}^1$ globally or on the (y, θ_2) torus locally. These wrapped branes will generically have KK modes from dimensional reduction along the compact (y, θ_2) direction. There have recently been proposals on decoupling the KK modes for dipole deformed theories. The works of Lunin-Maldacena and Gursoy-Nunez [35] have discussed the field theory living on D-branes when there is an NS flux with one leg on the brane and one leg orthogonal to the brane.

The conclusion for the solutions of [35] was that, by turning on the NS flux, the masses of the KK modes grow because the sizes of the S^2 or S^3 cycles decrease. Of course for our case there are no non-trivial three-cycles in the geometry. We have an explicit two-cycle and we know the local geometry around this cycle. A non-trivial three-cycle should appear after the transition; the geometry after the transition is not covered by the discussion in this paper. The only consistency condition which can be checked is that the size of the resolved two-cycle is indeed zero in the IR, as will be seen by the corresponding values of volumes with and without B-fields.

In our language, the proposal of [35] tells us that, by turning the NS field in the (y, θ_1) direction, the KK modes are the only sector charged under $U(1)_y \times U(1)_{\theta_2}$ where the gluons and the gluinos are chosen to not be charged under $U(1)_x$. The dipole deformation then appears only in the KK spectrum and does not change the four-dimensional field theory.

Now we can ask how the masses of the KK modes are changed in our case. In the (near)–local solution, the masses of the KK modes are inversely proportional to the area of the (quasi)–torus (y, θ_2). For constant values of y and θ_2, the area of the torus depends on the value of $\tau_{(2)}$ as well as the dz fibration. The results we want to compare are:
(a) The nonzero corrections to $\tau_{(2)}$ starting from $\mathcal{O}(r^3)$ for the non-deformed case[17].
(b) The non-zero corrections to $\tau_{(2)}$ starting from $\mathcal{O}(r^5)$ for the dipole deformed case.
(c) The non-zero corrections to the whole metric due to the underlying b field.
In the discussion below – to be presented soon – we will argue from this simple analysis that, near the local solution limit, the volume of the two-cycle on which we have wrapped $D5$ branes *decreases* in the dipole deformed theory. This would imply that the KK masses are indeed bigger in the dipole deformed theory and they can be decoupled from the QCD scale.

Coming back to the issues of warp factors and other things we now have to determine the functional forms of $f_i(r, \theta_i)$ for our case. As expected, the equations of motion for f_i are more complicated than (2.46) that we had earlier. We haven't been able to work out the full details, but an approximate equation can be given for f_i that relate the θ_i–variation of f_i to the warp factors that we had before, in the following way:

$$\frac{1}{\sqrt{1-b^2}} \frac{\partial f_i}{\partial \theta_i} + \frac{\alpha_6}{\mathcal{F}_6(r_0)} + \frac{f_i \cot \theta_i}{\sqrt{1-b^2}} \; = \; -\frac{2\beta_6}{\mathcal{F}_6(r_0)} \, r + \mathcal{O}(r^2) \tag{4.34}$$

[17] It could even be $\mathcal{O}(r^2)$ as all the warp factors are taken to be linear in r.

where $i = 1, 2$ and the $\sqrt{1-b^2}$ dependence above is only approximate and is valid near the point radially away from the chosen point (2.3).

The solution to the above equation is not difficult to find, and it is given by the following functional form that is a slight modification of what we had earlier in (2.48)[18]:

$$f_i \;=\; \sqrt{1-b^2}\left[\frac{\alpha_6}{\mathcal{F}_6(r_0)} + \frac{2\beta_6}{\mathcal{F}_6(r_0)}\, r + \mathcal{O}(r^2)\right] \cot\,\theta_i \tag{4.35}$$

This is again encouraging because the modification from (2.48) is very small. In fact in the far IR the metric fibration in (4.33) will be exactly the same as in (2.44) if we replace the Q in (2.47) by

$$Q \;=\; \frac{\alpha_6\sqrt{1-b^2}}{\mathcal{F}_6(r_0)}. \tag{4.36}$$

The above value of Q is not exact, as we have made some simplifying assumptions to get to this. How far this value of Q is away from the exact answer will be determined later in the paper. Our naive expectation would be to extrapolate the value of Q in (2.48) to the present case. This will tell us that we might be off by a quantity δQ where δQ is given by the following expression:

$$\delta Q \;=\; \frac{\left(1 + a_1 b^2 - \sqrt{1-b^2}\right)\alpha_6}{\mathcal{F}_6(r_0)}, \tag{4.37}$$

where a_1 is still an undetermined constant. Such a change will no doubt have an effect on the original equation (4.34) that determines the f_i as we shall discuss later, but we have reasons to believe that the r and θ_i dependence of $f_i(r, \theta_i)$ may still survive the correction proposed in (4.37) or its correct generalisation thereof to be presented later. This would mean that the metric with a b field deformation may take the final form

$$\begin{aligned} ds^2_{\mathcal{M}} \;&= \mathcal{F}(r)^2\; dr^2 + \kappa_o^{-1}\mathcal{F}(r)^{-2}\,\Big(dz + \Delta_1(b)\,\cot\,\theta_1\; dx + \Delta_2(b)\,\cot\,\theta_2\; dy + \mathcal{O}(r)\Big)^2 \;+ \\ &\quad + \mathcal{F}(r)\;|dz_1|^2 + \mathcal{F}(r)\;|dz_2|^2, \end{aligned} \tag{4.38}$$

where we have put in all the b field corrections in $\Delta_i(b)$ and r dependent corrections as $\mathcal{O}(r)$ in the dz–fibration. Observe also that we haven't yet determined the coefficients in the warp factor $\mathcal{F}$. These coefficients will eventually be related to some conserved charges

[18] Again for $\theta_i \neq 0$. For $\theta_i = 0$ the fibration becomes a total derivative.

in the system, but an exact determination of these – along the lines of [78] – will be postponed to future publications.

At this point we should put in the seven-branes. The seven-branes are not sources of B_{NS} fields, so we would expect the b field to remain unaffected. However string coupling would definitely be altered by the seven-branes. Without them the string coupling g_s is affected only by the background b field in the following way:

$$g_s^2 \; = \; \frac{(g_s^o)^2}{1 - \frac{a_1 b^2 \alpha_6}{\mathcal{F}_6(r_0)}\; r + \left[\frac{a_2 b^4 \alpha_6}{\mathcal{F}_6(r_0)} - \frac{a_1 b^2 \beta_6}{\mathcal{F}_6(r_0)} - \frac{a_1 b^2 \alpha_5 \alpha_6}{\mathcal{F}_5(r_0)\mathcal{F}_6(r_0)}\right] r^2 + \mathcal{O}(r^3)}, \tag{4.39}$$

where g_s^o is the string coupling in the absence of b field. We now need to see how the seven branes would modify the result. First, of course we have to figure out their back-reaction on the geometry (4.38). The generic ansatz for the metric of a seven-brane oriented along spacetime directions $x^{0,1,2,3}$ and at a point on the (x, θ_1) directions is given by

$$ds^2_{\rm D7} \; = \; h^{-1} \; ds^2_{\|} \; + \; h \; ds^2_{x\theta_1}, \tag{4.40}$$

where h is the so-called harmonic function. This would mean that the final metric with b fluxes, and seven-branes will be to modify (4.38) by the warp factor h in the following way:

$$\begin{aligned} ds^2_{\mathcal{M}} \; &= h(r)^{-1}\Big[ds^2_{0123} + \mathcal{F}(r)^2 \; dr^2 + \mathcal{F}(r) \; |dz_2|^2\Big] + h(r) \; \mathcal{F}(r) \; |dz_1|^2 \; + \\ &\quad + h(r)^{-1}\kappa_o^{-1}\mathcal{F}(r)^{-2} \; \Big(dz + \Delta_1(b) \cot \theta_1 \; dx + \Delta_2(b) \cot \theta_2 \; dy + \mathcal{O}(r)\Big)^2. \end{aligned} \tag{4.41}$$

This is almost the final metric that we want for our case because we expect switching on gauge fluxes on the D7 brane to get bound states of $D5$ branes will alter the above metric only by an additional warp factor[19]. The string coupling $\tilde{g}_s$ before switching on the gauge fluxes is easy to work out and is given by

$$\tilde{g}_s \; = \; \frac{g_s}{h^4} \tag{4.42}$$

where g_s is given by (4.39) above. All we need now to complete the story is to add gauge fluxes. Our initial analysis told us that the gauge fluxes giving rise to D5 branes

[19] The generic back-reactions are a little more involved than overall warp factor changes. However in certain special limits the back-reactions do simplify enough to show only overall changes in the warp factors [41].

charges can be evaluated as (2.52) assuming that the integral is over a finite sphere $\mathbf{P}^1$. Unfortunately this may not always be true, because our initial choice of metric given as (2.51) is *not* realised in the present set-up. In fact the metric on the D7 brane can be calculated precisely from (4.41) and is given at a point $x = x_0, \theta_1 = \theta_{10}$ by the following metric:

$$ds_s^2 = h^{-1}\left[ds^2_{0123} + \mathcal{F}^2\ dr^2 + \mathcal{F}\ |dz_2|^2 + \frac{1}{\kappa_o \mathcal{F}^2}\ \Big(dz + \Delta_2(b)\ \cot\ \theta_2\ dy\Big)^2\right]. \tag{4.43}$$

A careful look at the metric suggests something very interesting: the metric resembles closely a locally deformed Taub-NUT space! Therefore all the nice properties of a Taub-NUT space could presumably be applied here (with of course certain modifications to take into account the deformations[20]). In particular the metric (4.43) can give an answer to the puzzle that we raised above, namely, the non existence of a finite sized two-cycle. The metric (4.43) can in fact support *normalisable* anti-selfdual harmonic forms and therefore we can identify the gauge fluxes with these forms. A study of such harmonic forms has been done earlier in [79] and recently in [76] by considering various possible deformations of the Taub-NUT metric. However all the analysis done before were for globally deformed Taub-NUT. Here we have only the local version so to apply our techniques we have to assume that the (y, θ_2) torus will eventually become a sphere (or a squashed sphere) globally. This would mean that $\mathcal{F}^{-1}|dz_2|^2 \ \rightarrow\ r^2 d\Omega_2^2$ with Ω_2 being the metric of a (squashed) two-sphere globally. We now define a one-form on this space,

$$\zeta \ = \ g(r)\Big(dz + \Delta_2(b)\ \cot\ \theta_2\ dy\Big) \tag{4.44}$$

with one assumption: Δ_2 is only a function of the b field. The function $g(r)$ can be explicitly determined by considering the fact that the two form ω constructed out of this is anti-selfdual, and is given by

$$g(r) \ = \ \exp\left[-\Delta_2 \int \frac{dr}{r^2 \mathcal{F}^2 \sqrt{\kappa_o}}\right] \tag{4.45}$$

where the integral should be from any point r to $r \rightarrow \infty$ in the full global geometry. In the absence of a global picture, all we can confirm here is that $g(r)$ is a normalisable function.

[20] The warp factors $\mathcal{F}, h$ are not quite that of a Taub-NUT space, because we are viewing everything locally. But a slice of the metric does have a strong resemblance to a KK monopole.

This would then mean that we can define our gauge fluxes – which we will switch on the D7 brane – as

$$F \equiv N\omega = Nd\zeta \tag{4.46}$$

with ω normalisable but not globally defined; and N is an integer specifying the number of $D5$ branes. Once such fluxes are specified, the background RR field will be determined. Alternatively, we can assume that the warp factor $\mathcal{F}$ sufficiently curves the dz fibration to allow non-trivial two-cycles *a là* the two-cycles in the metric (2.51). In either case, bound states of D5 branes wrapped on two-cycles (y, θ_2) will be created. With this, the final metric with D5s, D7 and fluxes, is very close to the one presented above in (4.41) and is modified only by appropriate warp factor[21]. As we can clearly see now, near the point $r = r_0$ the final local metric is exactly the one that we had presented earlier in [26], [27] and [28]! This therefore serves as a very strong confirmation of our result.

One last step still remains: we need to determine the size of the two-cycle on which we have wrapped D5 branes. Now that we have an almost complete description of the local geometry, the metric of the two-cycle will not be too difficult to determine. For the constant value of (r, z, x, θ_1), the metric is diagonal and is given by

$$ds^2_{\rm two-cycle} = h^{-1}dy^2\left[\mathcal{F}(r_0) + \frac{\Delta_2^2 \cot^2\theta_2}{\kappa_o\,\mathcal{F}^2(r_0)}\right] + h^{-1}\,\mathcal{F}(r_0)\,d\theta_2^2 \tag{4.47}$$

where we see that the dz fibration also contributes to the metric of the two-cycle as one would have expected. Note also that the metric of the two-cycle depends on the b field from two different sources: Δ_2 and κ_o. Thus the volume of the two-cycle is given by

$$\text{Vol}_\text{b} = \int dy d\theta_2\,\frac{\mathcal{F}(r_0)}{h}\sqrt{1 + \frac{\Delta_2^2\cot^2\theta_2}{\kappa_o\,\mathcal{F}^3(r_0)}}, \tag{4.48}$$

where Vol_b denotes the volume calculated with back-reactions from the b field. In the absence of the b field we know that $\text{Vol}_0 = \int dyd\theta_2\,\frac{\mathcal{F}(r_0)}{h}\sqrt{1 + \frac{\cot^2\theta_2}{\mathcal{F}(r_0)^3}}$, and therefore the change in the volume is given by

$$\delta V = \text{Vol}_0 - \text{Vol}_\text{b} = \int dyd\theta_2\,\frac{\cot^2\theta_2}{2h\,\mathcal{F}^2(r_0)}\left(1 - \frac{\Delta_2^2}{\kappa_o}\right). \tag{4.49}$$

[21] There is a little more to it. Indeed the metric will pick up extra warp factors, but the background fluxes will also change. One can see this from an earlier analysis of [80] done for topologically trivial background geometry.

To see which volume is bigger we need the behavior of $\frac{\Delta_2^2}{\kappa_o}$ in terms of the order by order expansion that we have been using. Our earlier determination of (4.37) may simply suggest

$$\kappa_o \ < \ 1 \qquad \text{and} \qquad \Delta_2 \ < \ 1 \tag{4.50}$$

because $\mathcal{O}(r^{2n+1})$ terms for κ_o are dominant and contribute negatively to the sum in the series. Similarly, Δ_2 terms also seem to be suppressed as can be seen from (4.36). Unfortunately, these considerations do not help us to see the behavior of the ratio $\frac{\Delta_2^2}{\kappa_o}$ and therefore we need a more detailed analysis to figure this out.

Our first step would be to determine which of κ_o and Δ_2^2 is bigger. In our earlier order-by-order expansion, if we keep the expansion only to the first order in r, then one can show with some effort

$$\Delta_2^2 \ = \ 1 - a_1 b^2 \left(1 + \frac{\alpha_6}{\mathcal{F}_6(r_0)}\right) r_0 \tag{4.51}$$

where we have taken a point $r = r_0$ to do the analysis. From this, and using the expansion of κ_o, it is easy to show that

$$\kappa_o \ - \ \Delta_2^2 \ = \ a_1 b^2 \tag{4.52}$$

implying $\kappa_o > \Delta_2^2$, at least to the order that we have considered. This also means that

$$\text{Vol}_\text{b} \quad < \quad \text{Vol}_0 \tag{4.53}$$

and therefore the volume of the two-cycle decreases when we switch on a dipole deformation, again up to the orders that we have considered in our expansion. The question now is whether the result remains unchanged if we take higher order terms in our expansion series. For this we need to find a closed form for Δ_2^2. We therefore make the following observations:

• $\Delta_2^2 = 1$ when $\kappa_o = 1$. Furthermore Δ_2^2 can only be functions of κ_o and the warp factors $\mathcal{F}, h$ because these are the only unknown variables in our system.

• In the dual F-theory side $\Delta_2 \text{cot}\ \theta_i$ with $i = 1, 2$ appear explicitly as four-form G-fluxes. In M-theory these G-fluxes now couple not to the co-dimension four surfaces **A** and **B** (in fig 1), but to oriented co-dimension surfaces.

• The B field component that gave rise to the dipole deformation b picks up other additional components[22] – now parallel to the x and z directions – that are also proportional to Δ_2. Although this is a generic result, the new components are suppressed compared to the results that one would get without B fields.

The last observation is particularly useful to pin-point the precise value of Δ_2. In this paper we will not go through the analysis, as this could be derived easily from the inherent F-theory picture. The final result in compact form can be written as

$$\Delta_2^2 \;=\; \frac{\kappa_o\mathcal{F}-1}{\mathcal{F}-1} \tag{4.54}$$

which can now be easily shown to reproduce (4.52) and hence would finally explain the decoupling of the KK states.

Before finishing this section, we should evaluate the background H–fluxes. Since the dipole b–field cannot be gauged away, there must exist the corresponding H_{NS}. Furthermore this should correspond to our earlier expected ansatz (2.53). Now that we know most of the details regarding our background we should be able to verify this. Taking the metric factors correctly, we can show that locally there is indeed a H_{NS} given by

$$\mathcal{H} \;=\; \mathcal{H}_0 \;\mathrm{cosec}^2\;\theta_2 \; dy\wedge d\theta_2\wedge d\theta_1, \tag{4.55}$$

which is exactly of the form (2.53) as one might have expected[23]. The constant $\mathcal{H}_0$ could also be determined for our case once we put in the value of Δ_2 in (4.54). For the metric (4.38), $\mathcal{H}_0$ turns out to be

$$\mathcal{H}_0 \;=\; \frac{\sqrt{\mathcal{F}(1-\kappa_o)(\kappa_o\mathcal{F}-1)}}{\mathcal{F}-1} \tag{4.56}$$

where $\mathcal{F}$ and κ_o are all measured at $r = r_0$ and therefore $\mathcal{H}_0$ is a constant. Once we know H_{NS}, the H_{RR} – that forms the D5 sources – is easily determined from primitivity

[22] This would explain why one of the complex structure (4.18) in (4.17) is affected by the b-field. All the components of the b field have one leg along the θ_1 direction. Therefore the (x,θ_1) torus should definitely be affected. On the other hand due to (a) the fact that the other components of the b field are along all the isometry directions, and (b) the inherent gauge invariance of the b field, the (y,θ_2) torus is not affected but only the $U(1)$ fibration is, as apparent from (4.38).

[23] The above equation for $\mathcal{H}$ (4.55) is fine as long as we study far IR values. Due to the subtleties mentioned in [80], there will be other r dependent components. A full analysis of this will require more inputs than what we have mentioned here. These and other issues will be addressed in [41].

as shown earlier in (2.53). Along with the metric (4.41), string-coupling g_s (4.42) and the axion (which we leave for the readers to derive) the full background satisfying all the equations of motion can be completely determined.

5. Conclusions and future directions

In this paper we have addressed several issues concerning the generic realm of gauge-gravity dualities. Our first starting point was to clarify the fibration structure of the local metric that we have presented earlier in [26], [27] and [28]. A naive analysis would have led to a constant fibration of the form (2.2). That this is not the full story can only become apparent if we carefully consider the warp factors. With these considerations, our first result is the

- Metric given by (2.49) with non-trivial $U(1)$ fibration.

The metric clearly tells us how one should view the local geometry. It is interesting to note that there may exist a family of such solutions, all coming from different possible realisations of global geometries. The local solutions are supersymmetric, but their naive global extensions may not be supersymmetric. In fact in this paper we haven't been able to find a globally defined metric that forms the gravity dual of $\mathcal{N} = 1$ gauge theory with fundamental (and possibly bi-fundamental) flavors. Part of the reason lies in the UV picture which, for our case, is complicated by the presence of local and non-local seven branes, $D5$ branes and $D3$ branes. One thing is of course clear:

- The full background is conformally Kähler with fluxes and branes,

although it could be made non-Kähler using the underlying F-theory picture. We give example of all these cases by solving equations of motion order by order in powers of r. For a given patch, exemplified by fig. 3, the metric is well defined and the full global geometry would be to add up all the patches. From F-theory we know that the manifold has at least one $\mathbf{P}^1$ on which we have wrapped $D5$ branes. The local seven branes i.e the $D7$ branes when brought near the $D5$s would also wrap the $\mathbf{P}^1$. On an isolated $D7$ brane, the $D5$ branes could exist as bound states of $D5 - D7$. This is naturally supersymmetric and susy is only broken down to $\mathcal{N} = 1$ by the background geometry.

The story in the heterotic theory is more interesting. The background is not dual to the type IIB background, and therefore not constrained by the type IIB structure. Global metrics in heterotic theory do exist, and one example of this was already presented in [28].

In our language the global metric that we determined in [28] is what we called the metric after geometric-transition (GT). Our result therein was that

- The heterotic metric after GT was a warped version of a MN-type [24] metric.

This is of course for minimal susy. For $\mathcal{N} = 2$ the metric is given by a variant of the background of type [81]. On the other hand *before* GT the situation is more intriguing. The complete local background can be found for this case, and our results are

- The metric is given by (3.35) (or as (3.49) in a simplified form).
- The torsion is computed in terms of the torsion classes $\mathcal{W}_i$ in sec. 3.2.
- The complete mathematical structure of this manifold is given in sec. 3.3

For the mathematical parts, we have found a family of solutions given by holomorphic $\mathbf{C}^*$ fibrations arising from topologically nontrivial fibrations over the Kodaira surface $\mathbf{S}$. These manifolds are generically non-Kähler as can be seen from the non-zero values of $\mathcal{W}_{3,4,5}$. They are also complex because $\mathcal{W}_1 = \mathcal{W}_2 = 0$ can be imposed on the solutions. So we find

- New non-compact, non-Kähler complex manifolds in heterotic theory

whose local metric can be easily determined. The global story is another issue which we will dwell on in future works. The story, however, is not complete unless we figure out the vector bundles. In [28] we showed how vector bundles could be pulled through a conifold transition. A similar analysis should be done here because our manifold is the one *before* GT[24]. Additionally, it is also interesting to ask whether topologically non-trivial holomorphic $\mathbf{C}^*$ fibrations exist for arbitrary $\mathbf{S}$. In fact looking at the mathematics, it would seem that these rarely exist except for the cases where the Kodaira surface $\mathbf{S}$ has torsion[25] in H^2. The physical implication of this result is not clear to us at this stage.

[24] The local geometry that we analysed here has no holomorphic $\mathbf{P}^1$, only holomorphic T^2. This is of course similar to our earlier conclusion for the type IIB case. On the other hand the geometry of [28] is global and has non-trivial $\mathbf{S}^3$ on which we did the conifold transition.

[25] A brief explanation of how the torsion in $H^2(\mathbf{S}, \mathbf{Z})$ arises is as follows: Consider $\mathbf{S}$ as a T^2 fibration over $\mathbf{B} = T^2$. Each of the two S^1 fibrations has a chern class, which can be identified with an integer via $H^2(\mathbf{B}, \mathbf{Z}) = \mathbf{Z}$. If we call these integers r and s, then if r and s are relatively prime, then $H^2(\mathbf{S}, \mathbf{Z}) = \mathbf{Z}^4$. But if r and s have greatest common divisor $m > 1$, then $H^2(\mathbf{S}, \mathbf{Z}) = \mathbf{Z}^4 + \mathbf{Z}_m$.

Maybe this is restricting the choice of the intrinsic torsion in our framework, but we have no concrete conclusion on this right now.

For the type IIB theory we also have additional results. Once we know that the fibrations in the metric can be non-trivial, we can ask whether there could be other effects. One new effect could in principle come from the back-reactions of the B fields. For our case, due to the presence of branes and orientifold planes at the orientifold corner of the moduli space, the B fields backreact as **dipole** deformations in the field theory. Although the background geometry is complicated, we have been able to evaluate the local metric. Our results are

- The metric is given by (4.41).
- The three-form NS flux is given by (4.55).
- The three-from RR flux is given by the Hodge dual of (4.55), i.e (2.53).
- The string coupling is given by (4.42).

The five-form fluxes and the axion can be evaluated from above. Of course all these results would be further influenced by the subtleties mentioned in [80]. But these additional corrections would not be visible in the IR of the gauge theory *except* for one thing:

- Volume of the two-cycle *shrinks* due to the dipole deformation,

resulting in the KK modes – from the dimensional reduction on the two-cycle of the wrapped branes – becoming heavier. Therefore they can be integrated out from the gauge theory, giving rise to pure $\mathcal{N} = 1$ YM theory at the IR. Hence for all IR purposes our solutions for the background will be robust.

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